Ex 2: $T(n) = 5T\left(\frac{n}{4}\right) + n$

$a = 5$, $b = 4$, $\log_4 5 = 1.1609\ldots > 1$
LET $\varepsilon = \log_4 5 - 1$ SO $\log_4 5 - \varepsilon = 1$. THEN

$$f(n) = n^1 = O\left(n^{\log_4 5 - \varepsilon}\right)$$

$\therefore$ WE ARE IN CASE (1).

$$T(n) = \Theta\left(n^{\log_4 5}\right)$$

Ex 3: $T(n) = 5T\left(\frac{n}{4}\right) + n^2$

$a = 5$, $b = 4$, $\log_4 5 = 1.1609\ldots < 2$. LET $\varepsilon = 2 - \log_4 5$,
SO $\log_4 5 + \varepsilon = 2$. THEN

$$f(n) = n^2 = \Omega\left(n^{\log_4 5 + \varepsilon}\right)$$

$\therefore$ CASE (3). MUST CHECK REGULARITY CONDITION:

$$a f\left(\frac{n}{b}\right) \le c f(n) \quad \text{FOR SOME } c < 1.$$

i.e. $\frac{5}{16} n^2 \le c n^2$

WE MAY CHOOSE ANY c SATISFYING $\frac{5}{16} \le c < 1$.

SINCE THE REGULARITY CONDITION IS SATISFIED

$$T(n) = \Theta(n^2).$$

4.4 PROOF OF THE MASTER THEOREM.

WE PROVE THE MASTER THEOREM IN CASE (1) ONLY. SEE P. 79-84 FOR DETAILS ON OTHER CASES.

FOR DEFINITENESS, WE TAKE THE RECURRENCE TO BE

$$T(n) = \begin{cases} d & 1 \leq n < \lfloor b \rfloor \\ aT\left(\lfloor \frac{n}{b} \rfloor\right) + f(n) & n \geq \lfloor b \rfloor \end{cases}$$

THE PROOF IS SIMILAR IN THE CASE OF CEILINGS $T\left(\lceil \frac{n}{b} \rceil\right)$ OR FOR A DIFFERENT SET OF INITIAL TERMS, AND IS LEFT AS AN EXERCISE.

PROOF.

UPON ITERATING THE RECURRENCE WE OBTAIN

$$
\begin{aligned}
T(n) &= f(n) + aT\left(\left\lfloor \tfrac{n}{b} \right\rfloor\right) \\
&= f(n) + a\left(f\left(\left\lfloor \tfrac{n}{b} \right\rfloor\right) + aT\left(\left\lfloor \left\lfloor n/b \right\rfloor / b \right\rfloor\right)\right) \\
&= f(n) + a f\left(\left\lfloor \tfrac{n}{b} \right\rfloor\right) + a^2\, T\left(\left\lfloor \tfrac{n}{b^2} \right\rfloor\right) \\
&\;\;\vdots \\
&= \sum_{i=0}^{k-1} a^i f\left(\left\lfloor \tfrac{n}{b^i} \right\rfloor\right) + a^k\, T\left(\left\lfloor \tfrac{n}{b^k} \right\rfloor\right)
\end{aligned}
$$

THE PROCESS TERMINATES WHEN k IS CHOSEN SO THAT $1 \le \left\lfloor \frac{n}{b^k} \right\rfloor < \lfloor b \rfloor \le b$. i.e.

$$
\begin{aligned}
1 \le \left\lfloor \tfrac{n}{b^k} \right\rfloor < b &\iff 1 \le \tfrac{n}{b^k} < b \\
&\iff b^k \le n < b^{k+1} \\
&\iff k \le \log_b n < k+1 \\
&\iff k = \lfloor \log_b n \rfloor
\end{aligned}
$$

LET $k = \lfloor \log_b(n) \rfloor$ AND OBSERVE $k \le \log_b n$

THUS

$$(1) \quad T(n) = \sum_{i=0}^{k-1} a^i f(\lfloor n/b^i \rfloor) + d a^k$$

SINCE $f(n) = O(n^{\log_b a - \varepsilon})$ THERE EXISTS POSITIVE c AND n_0 SUCH THAT

$$(2) \quad 0 \le f(n) \le c n^{\log_b a - \varepsilon}$$

FOR ALL $n \ge n_0$.

OBSERVE THAT IF WE ALTER THE VALUE OF $f(n)$ FOR $n \in \{1, 2, \ldots, n_0 - 1\}$ THEN WE CHANGE A FIXED SET OF TERMS IN (1), WHICH DOES NOT CHANGE THE ASYMPTOTIC ORDER OF $T(n)$. THUS WE MAY ASSUME THAT (2) HOLDS FOR ALL $n \ge 1$.

THEREFORE FOR ALL $n \ge 1$:

$$0 \le f(\lfloor n/b^i \rfloor) \le c \lfloor n/b^i \rfloor^{\log_b a - \varepsilon} \le c (n/b^i)^{\log_b a - \varepsilon}$$

HENCE

$$\sum_{i=0}^{k-1} a^i f(\lfloor n/b^i \rfloor)$$

$$\le \sum_{i=0}^{k-1} a^i \cdot c \left(\frac{n}{b^i}\right)^{\log_b a - \varepsilon}$$

$$= c n^{\log_b a - \varepsilon} \cdot \sum_{i=0}^{k-1} a^i (b^{-i})^{\log_b a - \varepsilon}$$

$$= c n^{\log_b a - \varepsilon} \cdot \sum_{i=0}^{k-1} a^i (b^{\log_b a})^{-i} \cdot b^{i\varepsilon}$$

$$= c n^{\log_b a - \varepsilon} \cdot \sum_{i=0}^{k-1} (b^\varepsilon)^i$$

$$= c n^{\log_b a - \varepsilon} \cdot \left(\frac{(b^\varepsilon)^k - 1}{b^\varepsilon - 1}\right)$$

$$\le \left(\frac{c}{b^\varepsilon - 1}\right) n^{\log_b a - \varepsilon} \cdot \left[(b^{\log_b n})^\varepsilon - 1\right]$$

$$= \left(\frac{c}{b^\varepsilon - 1}\right)\left(n^{\log_b a} - n^{\log_b a - \varepsilon}\right)$$

$$= \Theta(n^{\log_b a})$$

Thus

$$T(n) \le \Theta(n^{\log_b a}) + d \cdot a^k$$

$$\le \Theta(n^{\log_b a}) + d\, a^{\log_b n} = O(n^{\log_b a})$$

$$\therefore T(n) = O(n^{\log_b a}).$$

BUT ALSO FROM (1) AND (2)

$$T(n) = \sum_{i=0}^{k-1} a^i f(\lfloor n/b^i \rfloor) + d a^k$$

$$\geq d a^k$$

$$\geq d a^{\log_b n - 1} \qquad (\lfloor x \rfloor > x - 1)$$

$$= \frac{d}{a} \cdot n^{\log_b a}$$

$$= \Omega(n^{\log_b a})$$

$\therefore \; T(n) = \Omega(n^{\log_b a})$ ALSO.

$\therefore \; T(n) = \Theta(n^{\log_b a})$ AS REQUIRED. ///.

READ PROOFS OF CASES (2) AND 3 IN BOOK.

CHANGE OF VARIABLES

Ex. $T(n) = T(\lfloor \sqrt{n} \rfloor) + 1$, $T(2) = T(3) = 0$

DEFINE $m = \lg n$. THEN $n = 2^m$ AND $\sqrt{n} = 2^{m/2}$.
NOW DEFINE

$$S(m) = T(2^m) = T(n) = T(\sqrt{n}) + 1 \quad (\text{DROPPING } \lfloor\ \rfloor)$$

$$= T(2^{m/2}) + 1 = S\left(\tfrac{m}{2}\right) + 1$$

$\therefore\ S(m) = S\left(\frac{m}{2}\right) + 1$

BY CASE 2 OF MASTER THEOREM,

$$S(m) = \Theta(\lg m)$$

THUS

$$T(n) = \Theta(\lg \lg (n))$$

EXERCISE:
FIND THE EXACT SOLUTION FOR $T(n)$.