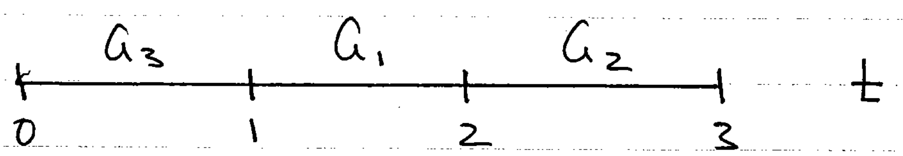
Scheduling Unit Time Tasks (16.5)

A <u>unit time task</u> is simply a job which requires one unit of time to complete.

Given a set $S = \{a_1, a_2, \ldots, a_n\}$ of n unit time tasks, a <u>schedule</u> for S is simply a permutation of S giving the order in which the tasks are to be performed. We assume that any schedule begins at time $t=0$ and ends at time $t=n$.

<u>Ex.</u> $S = \{a_1, a_2, a_3\}$ schedule: $a_3\, a_1\, a_2$



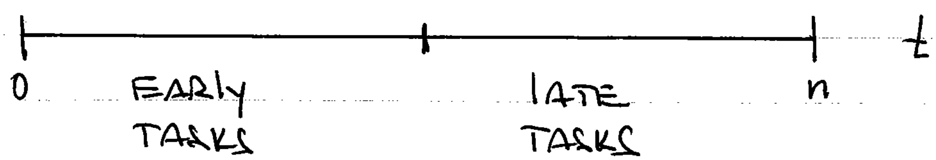
Suppose each task $a_i \in S$ has a deadline d_i satisfying $1 \le d_i \le n$, and a penalty $w_i \ge 0$ to be paid if a_i finishes later than its deadline $(1 \le i \le n)$.

<u>Problem</u>

Determine a schedule for S which minimizes the total penalty for missed deadlines.

Consider any schedule for S. We say a task is <u>early</u> in this schedule if it finishes before its deadline. Otherwise the task is said to be <u>late</u> in the schedule.

A schedule is said to be in <u>early-first form</u> if all early tasks are completed before any late tasks are started.



If some late task a_j is performed before some early task a_i, then upon swapping a_i with a_j, a_j is still late and a_i is still early. Thus any schedule can be placed in early-first form without changing it's penalty.

A schedule is said to be in <u>cannonical form</u> if early tasks precede late tasks, and the early tasks are scheduled in order of increasing deadlines.

ONE CAN GO FROM EARLY-FIRST FORM TO CANNONICAL FORM WITHOUT CHANGING THE PENALTY OF A SCHEDULE

PROOF

SUPPOSE a_i AND a_j ARE EARLY TASKS WHICH FINISH AT TIMES $t_i \le d_i$ AND $t_j \le d_j$ RESPECTIVELY, AND SUPPOSE ALSO THAT $t_j < t_i$ AND $d_i < d_j$.

UPON SWAPPING a_i WITH a_j IN THIS SCHEDULE WE SEE THAT a_j FINISHES AT TIME

$$t_j' = t_i \le d_i < d_j$$

AND a_i FINISHES AT TIME

$$t_i' = t_j < t_i \le d_i .$$

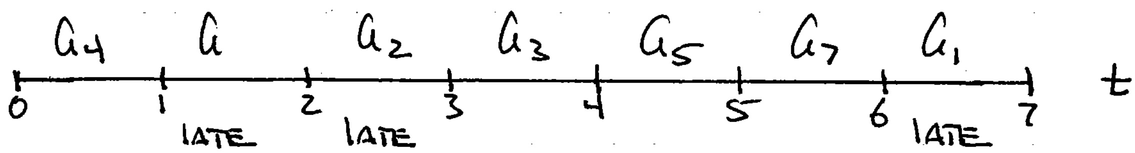
HENCE BOTH TASKS ARE EARLY IN THE NEW SCHEDULE, AND THE PENALTY FOR THE NEW SCHEDULE IS THE SAME AS FOR THE OLD.

CLEARLY ANY EARLY-FIRST SCHEDULE CAN BE PLACED IN CANNONICAL FORM BY PERFORMING SWAPS OF THIS KIND. ///

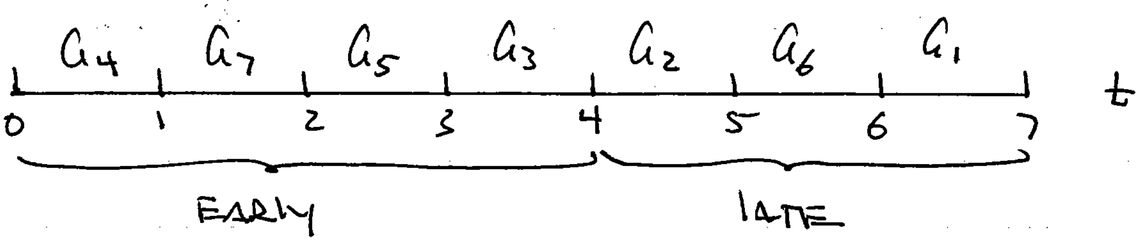
Ex. $n = 7$

TASK:	a_1	a_2	a_3	a_4	a_5	a_6	a_7
d_i:	4	1	5	2	6	1	6
w_i:	1	2	1	1	2	3	1

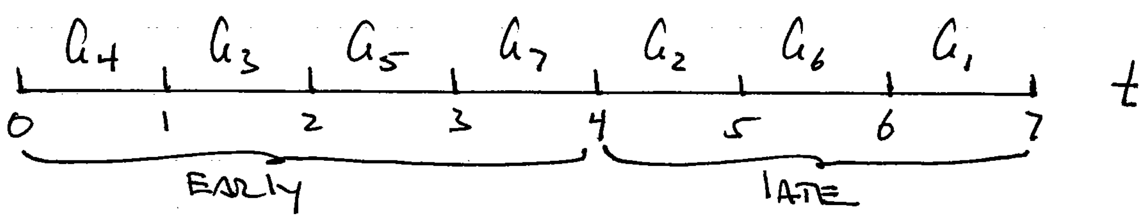
A SCHEDULE WITH PENALTY 6:



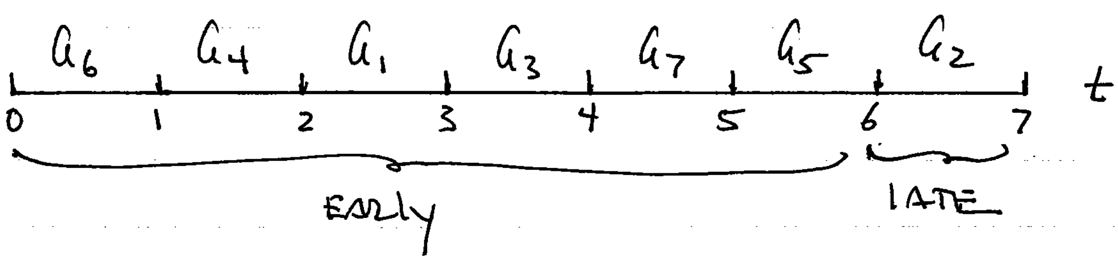
EARLY-FIRST FORM:



CANONICAL FORM:



AN OPTIMAL SCHEDULE (PENALTY = 2):



DEFINE

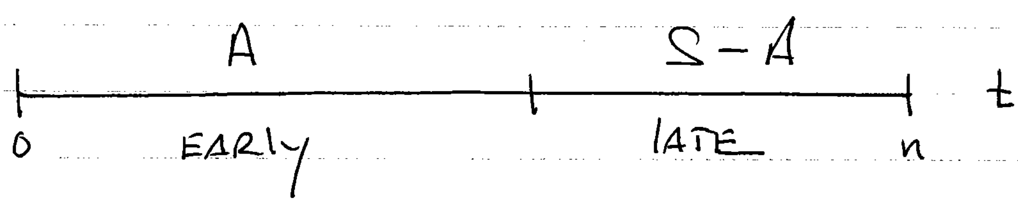
$$\mathcal{I} = \{ A \subseteq S \mid \text{THERE EXISTS A SCHEDULE FOR } S \text{ IN WHICH ALL TASKS IN } A \text{ ARE EARLY} \}.$$

OBVIOUSLY THE SET OF EARLY TASKS IN SOME SCHEDULE CONSTITUTES AN INDEPENDENT SET.

<u>THEOREM</u>

$M = (S, \mathcal{I})$ IS A MATROID.

OBSERVE THAT THE PROBLEM OF FINDING A SCHEDULE FOR S WHICH MINIMIZES THE PENALTY FOR MISSED DEADLINES IS EQUIVALENT TO FINDING A MAXIMUM WEIGHT INDEPENDENT SET IN THIS MATROID.



i.e. WE CAN MINIMIZE THE FINES WHICH MUST BE PAID BY MAXIMIZING THE FINES WHICH NEED NOT BE PAID.

DEFINE FOR $t=0,1,\ldots,n$ AND $A \subseteq S$:

$$N_t(A) = (\text{\# OF TASKS } a_i \in A \text{ SUCH THAT } d_i \le t)$$

NOTE THAT $N_0(A)=0$ AND $N_n(A)=|A|$ FOR ANY $A \subseteq S$.

LEMMA

LET $A \subseteq S$. THE FOLLOWING ARE EQUIVALENT.

(1) $A \in \mathcal{I}$

(2) $N_t(A) \le t$ FOR $t=0,1,\ldots,n$

(3) IF THE TASKS OF A ARE PERFORMED IN ORDER OF INCREASING DEADLINES (STARTING AT $t=0$ AND WITH NO IDLE TIME BETWEEN TASKS), THEN NO TASK IS LATE.

PROOF:

(1) $\Rightarrow$ (2)

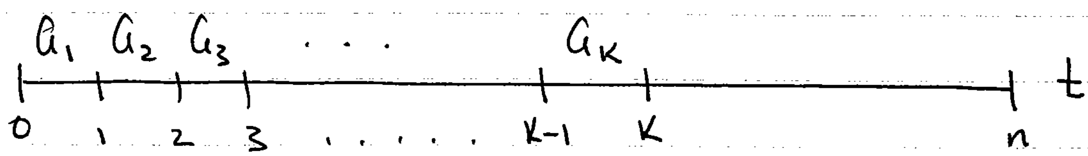
EQUIVALENTLY WE PROVE THE CONTRAPOSITIVE: NOT (2) $\Rightarrow$ NOT (1). SUPPOSE $N_t(A) > t$ FOR SOME t. THEN THERE ARE MORE THAN t TASKS IN A WHICH MUST FINISH BEFORE TIME t. THESE TASKS CANNOT BE SCHEDULED WITHOUT AT LEAST ONE OF THEM BEING LATE. $\therefore A \notin \mathcal{I}$.

$(2) \Rightarrow (3)$

ASSUME (2) HOLDS AND SUPPOSE THE TASKS IN A ARE SCHEDULED BY INCREASING DEADLINES. BY RE-INDEXING THE ELEMENTS OF S IF NECESSARY, WE MAY ASSUME

$$A = \{a_1, a_2, \ldots, a_K\} \subseteq S$$

WITH DEADLINES $d_1 < d_2 < \cdots < d_K$. OUR (PARTIAL) SCHEDULE IS THEN



SINCE $d_1 \geq 1$, TASK a_1 IS NOT LATE. BUT ALSO

$$N_1(A) \leq 1 \Rightarrow d_2 \geq 2 \Rightarrow a_2 \text{ NOT LATE}$$
$$N_2(A) \leq 2 \Rightarrow d_3 \geq 3 \Rightarrow a_3 \text{ NOT LATE}$$
$$N_3(A) \leq 3 \Rightarrow d_4 \geq 4 \Rightarrow a_4 \text{ NOT LATE}$$
$$\vdots$$
$$N_{K-1}(A) \leq K-1 \Rightarrow d_K \geq K \Rightarrow a_K \text{ NOT LATE}$$

$\therefore$ NO TASK IN A IS LATE.

$(3) \Rightarrow (1)$ IS OBVIOUS. ///

EXERCISE

WRITE AN ALGORITHM WHICH DETERMINES WHETHER OR NOT $A \subseteq S$ IS INDEPENDENT. (HINT: USE PART (2) OF THE PRECEDING LEMMA, AND RECALL COUNTING SORT.)

EXERCISE

WRITE AN ALGORITHM WHICH DETERMINES A SCHEDULE OF UNIT TIME TASKS WITH MINIMUM TOTAL PENALTY. (HINT: BASE YOUR ALGORITHM ON THE GREEDY ALGORITHM FOR WEIGHTED MATROIDS.)

IT REMAINS ONLY TO PROVE THAT $(S, \mathcal{I})$ IS A MATROID.

PROOF:

OBVIOUSLY S IS FINITE AND NON-EMPTY, AND $\mathcal{I}$ IS A COLLECTION OF SUBSETS OF S, SO THE FIRST AXIOM IS SATISFIED.

IF $B \subseteq A \in \mathcal{I}$, THEN THE SAME SCHEDULE IN WHICH THE TASKS OF A ARE EARLY ALSO HAS THE TASKS IN B EARLY SINCE $B \subseteq A$. THUS THE HEREDITARY PROPERTY IS SATISFIED.

TO PROVE THE EXCHANGE PROPERTY, LET $A, B \in \mathcal{I}$ WITH $|B| > |A|$. WE MUST SHOW B CONTAINS A TASK WHICH EXTENDS A.

DEFINE

$$K = \max\left\{ t \mid 0 \le t \le n \text{ AND } N_t(B) \le N_t(A) \right\}.$$

RECALL $N_0(B) = N_0(A) = 0$ SO THE ABOVE SET IS NON-EMPTY, WHENCE ITS MAXIMUM K EXISTS. ALSO NOTE

$$N_n(B) = |B| > |A| = N_n(A)$$

SO THAT $K < n$. THE DEFINITION OF K SAYS THAT $N_K(B) \le N_K(A)$ AND

$$N_t(B) > N_t(A) \quad \text{FOR} \quad K < t \le n.$$

IN PARTICULAR

$$N_{K+1}(B) > N_{K+1}(A).$$

THUS

$$N_{K+1}(B) - N_K(B) > N_{K+1}(A) - N_K(A).$$

THIS LAST INEQUALITY SAYS THAT B CONTAINS MORE TASKS WITH DEADLINE K+1 THAN DOES A.

LET $a_i \in B - A$ WITH $i = K+1$, AND DEFINE

$$A' = A \cup \{a_i\}.$$

WE USE PART (2) OF THE PRECEDING LEMMA TO SHOW $A' \in \mathrm{II}$.

BY THE DEFINITION OF A' WE HAVE $N_t(A') = N_t(A)$ FOR $0 \leq t \leq K$, AND SINCE $A \in \mathrm{II}$ WE HAVE $N_t(A) \leq t$. THUS

$$N_t(A') \leq t \quad \text{FOR} \quad 0 \leq t \leq K.$$

AGAIN BY THE DEFINITION OF A' WE HAVE $N_t(A') \leq N_t(A) + 1$ FOR ANY t. BUT RECALL $K < t \leq n$ IMPLIES $N_t(A) < N_t(B)$ WHENCE $N_t(A) + 1 \leq N_t(B)$. ALSO $N_t(B) \leq t$ SINCE $B \in \mathrm{II}$. THUS

$$N_t(A') \leq t \quad \text{FOR} \quad K < t \leq n.$$

$\therefore A' \in \mathrm{II}$ AS REQUIRED. ///