IT IS SOMETIMES POSSIBLE TO DO THESE CALCULATIONS WITH A LITTLE MORE RIGOR AND OBTAIN AN ASYMPTOTIC SOLUTION DIRECTLY. WE CALL THIS THE <u>ITERATION METHOD</u>.

USING THE SAME EXAMPLE:

$$T(n) = \begin{cases} 1 & 1 \le n < 3 \\ 2T(\lfloor n/3 \rfloor) + n & n \ge 3 \end{cases}$$

WE NO LONGER ASSUME THAT $n$ IS AN EXACT POWER OF 3. WE SUBSTITUTE THIS EXPRESSION INTO ITSELF:

$$\begin{aligned} T(n) &= n + 2T(\lfloor n/3 \rfloor) \\ &= n + 2\left(\lfloor n/3 \rfloor + 2T\left(\left\lfloor \lfloor n/3 \rfloor / 3 \right\rfloor\right)\right) \\ &= n + 2\lfloor n/3 \rfloor + 2^2 T(\lfloor n/3^2 \rfloor) \\ &= n + 2\lfloor n/3 \rfloor + 2^2\left(\lfloor n/3^2 \rfloor + 2T\left(\left\lfloor \lfloor n/3^2 \rfloor / 3 \right\rfloor\right)\right) \\ &= n + 2\lfloor n/3 \rfloor + 2^2\lfloor n/3^2 \rfloor + 2^3 T(\lfloor n/3^3 \rfloor) \\ &\vdots \\ &= \sum_{i=0}^{k-1} 2^i \lfloor n/3^i \rfloor + 2^k T(\lfloor n/3^k \rfloor) \end{aligned}$$

WE CHOOSE K SO THAT :

$$1 \le \lfloor n/3^K \rfloor < 3$$

$$\therefore \quad 1 \le n/3^K < 3$$

$$\therefore \quad 3^K \le n < 3^{K+1}$$

$$\therefore \quad K \le \log_3 n < K+1$$

$$\therefore \quad K = \lfloor \log_3(n) \rfloor .$$

WITH THIS VALUE OF K WE HAVE

$$T(n) = \sum_{i=0}^{K-1} 2^i \lfloor n/3^i \rfloor + 2^K$$

$$\le n \left( \sum_{i=0}^{K-1} \left(\tfrac{2}{3}\right)^i \right) + 2^{\log_3 n}$$

$$= n \left( \frac{1-\left(\frac{2}{3}\right)^K}{1-\left(\frac{2}{3}\right)} \right) + n^{\log_3 2}$$

$$= 3n \left( 1 - \left(\tfrac{2}{3}\right)^K \right) + n^{\log_3 2}$$

$$\le 3n + n^{\log_3 n} = O(n)$$

$\therefore$ $T(n) = O(n)$ HAS BEEN PROVED. ALSO

$$T(n) = \sum_{i=0}^{K-1} 2^i \lfloor n/3^i \rfloor + 2^K$$

$$\ge \sum_{i=0}^{K-1} 2^i \left( \frac{n}{3^i} - 1 \right) + 2^K$$

$$= n\left(\sum_{i=0}^{n}\left(\tfrac{2}{3}\right)^i\right) - \sum_{i=0}^{K-1} 2^i + 2^K$$

$$\geq n - \left(\frac{2^K-1}{2-1}\right) + 2^K$$

$$= n+1-2^K+2^K$$

$$= n+1 = \Omega(n).$$

$\therefore\ T(n) = \Omega(n)$, WHENCE $T(n) = \Theta(n)$.

THE ITERATION METHOD CAN SOMETIMES BE USED TO FIND EXACT SOLUTIONS TO RECURRENCE RELATIONS (AS OPPOSED TO ASYMPTOTIC SOLUTIONS.)

<u>EX.</u>

$$T(n) = \begin{cases} 3 & 0 \leq n < 2 \\ T(n-2)+n & n \geq 2 \end{cases}$$

$$\begin{aligned} T(n) &= n + T(n-2) \\ &= n + (n-2) + T(n-4) \\ &= n + (n-2) + (n-4) + T(n-6) \\ &\ \vdots \\ &= \sum_{i=0}^{K-1}(n-2i) + T(n-2K) \end{aligned}$$

CHOOSE $K$ SO THAT

$$0 \le n - 2K < 2$$
$$\therefore \quad 2K \le n < 2K+2$$
$$\therefore \quad K \le n/2 < K+1$$
$$\therefore \quad K = \lfloor n/2 \rfloor .$$

THEN $T(n-2K) = 3$, AND

$$T(n) = n\left(\sum_{i=0}^{K-1} 1\right) - 2\left(\sum_{i=0}^{K-1} i\right) + 3$$
$$= Kn - 2\left(\frac{K(K-1)}{2}\right) + 3$$
$$= Kn - K(K-1) + 3 ,$$

WHENCE

$$T(n) = \lfloor \tfrac{n}{2} \rfloor \cdot n - \lfloor \tfrac{n}{2} \rfloor \left(\lfloor \tfrac{n}{2} \rfloor - 1\right) + 3$$

EXERCISE

(1) $\lfloor f(n) \rfloor = \Theta(f(n))$

(2) $\Theta(f(n)) \cdot \Theta(g(n)) = \Theta(f(n) \cdot g(n))$.

IT FOLLOWS THAT $T(n) = \Theta(n^2)$. ONE CAN ALSO SHOW BY SOME ALGEBRA THAT

$$T(n) = 3 + \frac{n(n+2)}{4} + \frac{3}{8}\left((-1)^n - 1\right) .$$

## 4.3 THE MASTER THEOREM

THIS IS A METHOD FOR FINDING (ASYMPTOTIC) SOLUTIONS TO RECURRENCES OF THE FORM

(*) $$T(n) = a\,T\left(\frac{n}{b}\right) + f(n)$$

WHERE $a \geq 1$, $b > 1$ AND $f(n)$ IS ASYMPTOTICALLY POSITIVE. HERE $T\left(\frac{n}{b}\right)$ DENOTES EITHER $T\left(\lfloor \frac{n}{b} \rfloor\right)$ OR $T\left(\lceil \frac{n}{b} \rceil\right)$, AND IT IS UNDERSTOOD THAT $T(n) = \Theta(1)$ FOR SOME FINITE SET OF INITIAL TERMS.

SUCH A RECURRENCE DESCRIBES THE RUNNING TIME OF A "DIVIDE AND CONQUER" ALGORITHM WHICH DIVIDES A PROBLEM OF SIZE $n$ INTO $a$ SUBPROBLEMS, EACH OF SIZE $n/b$. HERE $f(n)$ REPRESENTS THE COST OF DOING THE DIVIDING AND RE-COMBINING.

**MASTER THEOREM**

LET $a \geq 1$, $b > 1$, $f(n)$ ASYMPTOTICALLY POSITIVE, AND LET $T(n)$ BE DEFINED BY (*). THEN

(1) IF $f(n) = O\left(n^{\log_b a - \varepsilon}\right)$ FOR SOME $\varepsilon > 0$ THEN

$$T(n) = \Theta\left(n^{\log_b a}\right).$$

(2) IF $f(n) = \Theta(n^{\log_b a})$, THEN

$$T(n) = \Theta(n^{\log_b(a)} \cdot \lg(n))$$

(3) IF $f(n) = \Omega(n^{\log_b a + \varepsilon})$ FOR SOME $\varepsilon > 0$,
AND IF $a f(\frac{n}{b}) \le c f(n)$ FOR SOME $c < 1$
AND SUFFICIENTLY LARGE $n$, THEN

$$T(n) = \Theta(f(n)).$$

REMARKS

IN EACH CASE WE COMPARE $f(n)$ TO THE FUNCTION $n^{\log_b a}$, AND THE SOLUTION IS DETERMINED BY WHICH IS LARGER.

IN CASE (1) $n^{\log_b a}$ IS POLYNOMIALLY LARGER AND THE SOLUTION IS $\Theta(n^{\log_b a})$. IN CASE (3) $f(n)$ IS LARGER (AND AN ADDITIONAL REGULARITY CONDITION IS MET) AND THE SOLUTION IS $\Theta(f(n))$. IN CASE (2) THE TWO FUNCTIONS ARE ASYMPTOTICALLY EQUIVALENT AND THE SOLUTION IS $\Theta(f(n) \cdot \lg n)$.

TO SAY THAT $f(n)$ IS POLYNOMIALLY SMALLER THAN $n^{\log_b a}$ (AS IN (1)) MEANS THAT THE TWO FUNCTIONS DIFFER BY A FACTOR $n^{\varepsilon}$ FOR SOME $\varepsilon > 0$.

NOTE THAT THE THREE CASES DO NOT COVER ALL POSSIBILITIES. THERE IS A GAP BETWEEN CASES (1) AND (2) WHEN $f(n)$ IS SMALLER THAN $n^{\log_b a}$ BUT NOT POLYNOMIALLY SMALLER.

EX. $T(n) = 2T\left(\frac{n}{2}\right) + \frac{n}{\lg n}$

$\log_2 2 = 1$, $\frac{n}{\lg n} \neq O(n^{1-\varepsilon})$ FOR ANY $\varepsilon > 0$.

SIMILAR COMMENTS HOLD FOR CASES (2) AND (3). IN ADDITION, THE REGULARITY CONDITION IN (3) MAY FAIL TO HOLD.

EXAMPLES

EX. 1: $T(n) = 8T\left(\frac{n}{2}\right) + n^3$

$a = 8$, $b = 2$, $\log_b a = 3$, $f(n) = n^3 = \Theta(n^{\log_b a})$

$\therefore$ CASE (2).

$$\therefore T(n) = \Theta(n^3 \lg(n)).$$