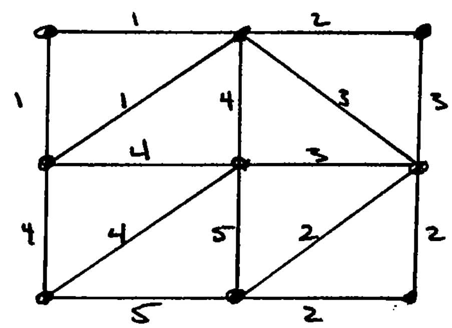
SUPPOSE $G = (V, E)$ IS EQUIPPED WITH A NON-NEGATIVE WEIGHT FUNCTION ON EDGES

$$w : E \longrightarrow \mathbb{R}_+$$

EX.



- THE WEIGHT OF A SPANNING TREE IS THE SUM OF THE WEIGHTS OF ITS EDGES.

PROBLEM

DETERMINE A SPANNING TREE OF MINIMUM WEIGHT IN G.

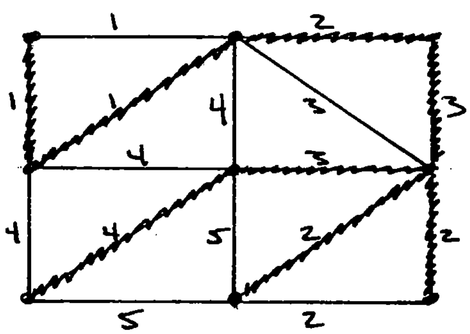
WE EXAMINE TWO FAMOUS GREEDY ALGORITHMS WHICH SOLVE THIS PROBLEM.

IN WHAT FOLLOWS LET $G = (V, E)$, $|V| = n$.

Prim's Algorithm (23.2)

- Choose an initial vertex (which is a tree)
- Amongst all edges incident with the current tree whose addition would not create a cycle, choose one of minimum weight.
- Stop when n-1 edges have been selected.

Ex



$W(T) = 18$

Observe that at each stage of execution, Prim's algorithm maintains a tree since no cycles are created and only incident edges are added.

When this tree contains n-1 edges it must have n vertices (by previous theorem), hence it is a spanning tree.

Theorem

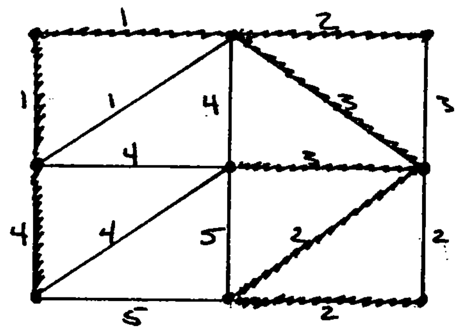
This spanning tree has minimum possible weight.

(See book or take CE 177 for proof.)

KRUSKAL'S ALGORITHM (23.2)

- CHOOSE AN EDGE OF MINIMUM WEIGHT
- AMONGST ALL EDGES WHICH DO NOT CREATE A CYCLE WITH PREVIOUSLY SELECTED EDGES, CHOOSE ONE OF MINIMUM WEIGHT.
- STOP WHEN $n-1$ EDGES HAVE BEEN SELECTED.

EX.



$w(T) = 18$

OBSERVE THAT AT EACH STAGE OF EXECUTION KRUSKAL'S ALGORITHM HAS CREATED A FOREST (UNION OF DISJOINT SUBTREES) SINCE NO CYCLES ARE CREATED.

WHEN THIS FOREST CONTAINS $n-1$ EDGES IT MUST ALSO HAVE n VERTICES. (ANY ~~GRAPH~~ FOREST WITH $n-1$ EDGES HAS AT LEAST n VERTICES. THIS FOREST CAN CONTAIN NO MORE THAN n VERTICES SINCE IT IS A SUBGRAPH OF G.)

THUS THE RESULTING FOREST IS CONNECTED (BY PREVIOUS THEOREM) AND IS A SPANNING TREE IN G.

THEOREM

THIS SPANNING TREE HAS MINIMUM WEIGHT AMONG ALL SPANNING TREES IN G.

PROOF.

LET T BE THE SPANNING TREE IN G CREATED BY KRUSKAL'S ALGORITHM, AND LET S BE ANY OTHER SPANNING TREE. WE MUST SHOW

$$w(T) \leq w(S)$$

LET $e_1, e_2, .. e_{n-1}$ BE THE EDGES OF T IN THE ORDER SELECTED BY KRUSKAL'S ALGORITHM. SINCE $S \neq T$ THERE IS A FIRST EDGE e_K WHICH IS NOT IN S, i.e.

$$\{e_1, .., e_{K-1}\} \subseteq E(S)$$
$$e_K \notin E(S)$$

LET H BE THE SUBGRAPH OBTAINED BY ADDING e_K TO S : $H = S + e_K$. BY THE TREENESS THEOREM H CONTAINS A UNIQUE CYCLE WHICH INCLUDES e_K, CALL IT C. NOTE C MUST CONTAIN AN EDGE e OF S WHICH IS NOT IN T, FOR OTHERWISE C IS CONTAINED IN THE ACYCLIC T.