CS 201: Quick reference guide to solving recurrences

The recipes in this handout for solving linear recurrences can be found in [BB88]. The first section deals with linear recurrences and the other two sections present techniques which can often be used to transform non-linear recurrences into linear ones. Additional references on solving recurrences and the derivation of the characteristic equation are also given.

Linear Recurrences

In this section three cases are presented, each more general than the preceding case.

Case 1

Suppose first that you are given a recurrence of the form,

$$T(n) = b_1 T(n-1) + b_2 T(n-2) + \ldots + b_k T(n-k) \tag{1}$$

with k initial conditions(i.e. values for $T(0), \ldots, T(k-1)$).

The *characteristic equation* for this recurrence is,

$$x^k - b_1 x^{k-1} - b_2 x^{k-2} - \ldots - b_k = 0. \tag{2}$$

If $r_1, r_2, \ldots, r_k$ are distinct roots of the characteristic equation, then the recurrence has a solution of the form,

$$T(n) = c_1 \cdot r_1^n + c_2 \cdot r_2^n + \ldots + c_k \cdot r_k^n$$

for some choice of constants $c_1, c_2, \ldots, c_k$. These constants can be determined from the k initial conditions.

Example 1:

$$T(n) = 7T(n-1) - 6T(n-2), \quad T(0) = 2,\ T(1) = 7.$$

Characteristic equation:

$$x^2 - 7x + 6 = (x-6)(x-1) = 0$$

General form of solution:

$$T(n) = c_1 6^n + c_2 (1)^n = c_1 6^n + c_2$$

Constraints for constants:

$$\begin{aligned} T(0) = \quad 2 &\quad = c_1 + c_2 \\ T(1) = \quad 7 &\quad = 6c_1 + c_2. \end{aligned}$$

Solution for constants: $c_1 = 1$ and $c_2 = 1$.

Solution: $T(n) = 6^n + 1$. □

Case 2

When the roots of the characteristic equation (Equation 2) are not distinct, then the general form of the solution includes the next highest power of n every time a root is repeated. That is, if there are h distinct roots $r_1, r_2, \ldots, r_h$ and m_i is the multiplicity of the i^{th} root, r_i, then the solution has the form

$$T(n) = \sum_{j=1}^{m_1} c_{1,j} n^{j-1} r_1^n + \sum_{j=1}^{m_2} c_{2,j} n^{j-1} r_2^n + \ldots + \sum_{j=1}^{m_i} c_{i,j} n^{j-1} r_i^n + \ldots + \sum_{j=1}^{m_h} c_{k,j} n^{j-1} r_h^n.$$

There are $m_1 + m_2 + \ldots + m_h = k$ constants which can again be determined from the k initial conditions.

Example 2:

$$T(n) = 3T(n-1) - 4T(n-3), \quad T(0) = -4,\ T(1) = 2,\ T(2) = 6.$$

Characteristic equation:

$$x^3 - 3x^2 + 4 = (x+1)(x-2)^2 = 0$$

General form of solution:

$$T(n) = c_1(-1)^n + c_2 2^n + c_3 n 2^n$$

Constraints for constants:

$$\begin{aligned} T(0) = &\quad -4 &&= c_1 + c_2 \\ T(1) = &\quad 2 &&= -c_1 + 2c_2 + 2c_3 \\ T(2) = &\quad 6 &&= c_1 + 4c_2 + 8c_3. \end{aligned}$$

Solution for constants: $c_1 = -2$, $c_2 = -2$ and $c_3 = 2$.

Solution: $T(n) = -2 \cdot (-1)^n - 2 \cdot 2^n + 2n \cdot 2^n = -2 \cdot (-1)^n + 2(n-1)2^n$. □

Case 3

If the recurrence in Equation 1 contains some terms of the form $e^n p(n)$ where e is a constant[1] and $p(n)$ is a polynomial in n, then additional roots are added to the characteristic equation. Specifically, when the recurrence is of the form

$$T(n) = b_1 T(n-1) + b_2 T(n-2) + \ldots + b_k T(n-k) + e_1^n p_1(n) + e_2^n p_2(n) + \ldots + e_s^n p_s(n),$$

where $e_1, e_2, \ldots, e_s$ are constants and $p_1, p_2, \ldots, p_s$ are polynomials in n, then the characteristic equation is

$$(x^k - b_1 x^{k-1} - b_2 x^{k-2} - \ldots - b_k)(x - e_1)^{d_1+1}(x - e_2)^{d_2+1} \ldots (x - e_s)^{d_s+1} = 0.$$

Here d_i if the degree of $p_i(n)$. The general form of the solution is the same as in the previous case, however there are now more than k constants to resolve. The additional $d_1 + 1 + d_2 + 1 + \ldots + d_s + 1$

[1]not necessarily 2.7182818 ...

initial values needed to resolve these constants can be obtained by plugging into the recurrence. This is how the coefficients of the polynomials in the recurrence influence its solution.

Example 3:

$$T(n) = 2T(n-1) + n + 2^{n+1}, \quad T(0) = 0.$$

The two extra terms in the form $e^n p(n)$ are $1^n(n)$ and $2^n(2)$. Characteristic equation:

$$(x-2)(x-1)^{1+1}(x-2)^{0+1} = (x-1)^2(x-2)^2 = 0$$

General form of solution:

$$T(n) = c_1 1^n + c_2 n 1^n + c_3 2^n + c_4 n 2^n$$

Three additional values can be obtained from the recurrence: $T(1) = 5,\ T(2) = 20,\ T(3) = 59$. Constraints for constants:

$$\begin{aligned} T(0) = &\quad 0 &= c_1 + c_3 \\ T(1) = &\quad 5 &= c_1 + c_2 + 2c_3 + 2c_4 \\ T(2) = &\quad 20 &= c_1 + 2c_2 + 4c_3 + 8c_4 \\ T(3) = &\quad 59 &= c_1 + 3c_2 + 8c_3 + 24c_4. \end{aligned}$$

Solution for constants: $c_1 = -2$, $c_2 = -1$, $c_3 = 2$ and $c_4 = 2$.

Solution: $T(n) = -2 - n + 2^{n+1}(n+1)$. □

Domain Transformations aka Change of Variable

A recurrence relation for $T(n)$ might involve previous terms which are not within a fixed range of n. Domain transformations can sometimes be used to substitute a function for the argument of the relation and remedy this situation. Specifically, given a recurrence relation for $T(n)$ in terms of $T(f(n)),\ T(f(f(n))),\ \ldots,\ T(f^k(n))$, the idea is to select a function, $g(m)$, with the property that $f(g(m)) = g(m-1)$. Substituting $g(m)$ for n gives a recurrence relation for $T(g(m))$ in terms of $T(f(g(m))),\ T(f(f(g(m)))),\ \ldots,\ T(f^k(g(m)))$ and these terms can be rewritten as $T(g(m-1)),\ T(g(m-2)),\ \ldots,\ T(g(m-k))$. A linear recurrence relation for $S(m)$ in terms of $S(m-1), S(m-2), \ldots, S(m-k)$ is obtained by equating $S(m)$ with $T(g(m))$. A solution for $S(m)$, can be transformed back into a solution for $T(n)$ by substituting $g^{-1}(n)$ for m. That is, since $T(g(m)) = S(m)$, $T(n) = S(g^{-1}(n))$.

Example 4:

$$T(n) = 7T(\frac{n}{2}) - 6T(\frac{n}{4}), \quad T(1) = 2,\ T(2) = 7.$$

In this case $f(n) = \frac{n}{2}$ and by selecting $g(m) = 2^m$ we obtain

$$f(g(m)) = \frac{g(m)}{2} = \frac{2^m}{2} = 2^{m-1} = g(m-1).$$

By equating n with 2^m we obtain

$$T(2^m) = 7T(\frac{2^m}{2}) - 6T(\frac{2^m}{4}) = 7T(2^{m-1}) - 6T(2^{m-2}).$$

Let $S(m) = T(2^m)$. The recurrence for $S(m)$ is

$$S(m) = 7S(m-1) - 6S(m-2), \quad S(0) = T(1) = 2,\ S(1) = T(2) = 7.$$

Note that the initial conditions for $S(m)$ were also transformed. This recurrence was solved in Example 1; its solution is $S(m) = 6^m + 1$. The inverse of $g(m) = 2^m$ is $g^{-1}(n) = \log_2 n$. The solution for $T(n)$ is

$$T(n) = S(g^{-1}(n)) = S(\log_2 n) = 6^{\log_2 n} + 1 = n^{\log_2 6} + 1.$$

□

Range Transformations

A recurrence relation may not always be linear. For example, there might be a product of terms in the relation. A range transformation attempts to apply a function to the recurrence to make it linear. Specifically, given a recurrence relation for $T(n)$ in terms of $T(n-1), T(n-2), \ldots, T(n-k)$, the idea is to find a function $f(x)$ such that $f(T(n))$ is a linear combination of $f(T(n-1)), f(T(n-2)), \ldots, f(T(n-k))$. Then by equating $W(n)$ with $f(T(n))$ we obtain a linear recurrence relation which can be solved using the methods in the first section. Since $T(n)$ is $f^{-1}(W(n))$ a solution for $W(n)$ can be transformed back into a solution for $T(n)$.

Example 5:

$$T(n) = 2 \cdot \frac{T(n-1)^3}{T(n-2)^2}, \quad T(0) = 2,\ T(1) = 2.$$

In this case by selecting $f(x) = \log_2 x$ we obtain

$$\begin{aligned} f(T(n)) &= \log_2 T(n) \\ &= \log_2 2 + \log_2(T(n-1)^3) - \log_2 T(n-2)^2 \\ &= 3\log_2 T(n-1) - 2\log_2 T(n-2) + 1 \\ &= 3f(T(n-1)) - 2f(T(n-2)) + 1. \end{aligned}$$

If we let $W(n) = f(T(n))$ and the we obtain the following recurrence for $W(n)$

$$W(n) = 3W(n-1) - 2W(n-2) + 1,$$

which has characteristic equation:

$$(x-1)(x-2)(x-1)^{0+1} = (x-1)^2(x-2) = 0.$$

The general form of the solution is:

$$W(n) = c_1 1^n + c_2 n 1^n + c_3 2^n.$$

Since there are three constants, one additional initial value must be obtained from the recurrence for solving the constants:

$$T(2) = 2 \cdot \frac{T(1)^3}{T(0)^2} = 2\frac{2^3}{2^2} = 4.$$

The values for the first three terms of $T(n)$ must be transformed into the corresponding values for $W(n)$ using $f()$.

$$\begin{aligned} W(0) = f(T(0)) = \log_2 T(0) &= \log_2 2 = 1 \\ W(1) = f(T(1)) = \log_2 T(1) &= \log_2 2 = 1 \\ W(2) = f(T(2)) = \log_2 T(2) &= \log_2 4 = 2 \end{aligned}$$

Constraints for constants:

$$\begin{aligned} W(0) = &\quad 1 &= c_1 + c_3 \\ W(1) = &\quad 1 &= c_1 + c_2 + 2c_3 \\ W(2) = &\quad 2 &= c_1 + 2c_2 + 4c_3 \end{aligned}$$

Solution for constants: $c_1 = 0$, $c_2 = -1$, and $c_3 = 1$.

The solution to $W(n)$'s recurrence is $2^n - n$.

The final step is to transform the solution for $W(n)$ into the solution for $T(n)$ using the inverse of $f()$. The inverse of $f(x) = \log_2 x$ is $f^{-1}(y) = 2^y$ so the solution for $T(n)$ is

$$T(n) = f^{-1}(W(n)) = 2^{W(n)} = 2^{2^n - n} = \frac{2^{2^n}}{2^n}.$$

□

References

[BB88] Gilles Brassard and Paul Bratley. *Algorithmics, Theory and Practice.* Prentice Hall, Englewood Cliffs, New Jersey, 1988.

[GK81] Daniel H. Greene and Donald E. Knuth. *Mathematics for the Analysis of Algorithms.* Birkhauser, Boston, Massachusetts, 1981.

[GKP89] Ronald L. Graham, Donald E. Knuth, and Oren Patashnik. *Concrete Mathematics.* Addison Wesley, Reading, Massachusetts, 1989.

[Wil90] Herbert S. Wilf. *generatingfuntionology.* Academic Press, San Diego, California, 1990.