FINAL

CIS 132 - Winter 02

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This exam is closed book and closed notes.
Show partial solutions to get partial credit.
If your questions are not written legibly, you won't get full credit.
Clarity and succinctness will be rewarded!

question 1:____________(out of 20)
question 2:____________(out of 10)
question 3:____________(out of 15)
question 4:____________(out of 10)
question 5:____________(out of 15)
question 6:____________(out of 10)
question 7:____________(out of 10)
question 8:____________(out of 10)

Total:____________(out of 100)

1. **a)** Let H be the language associated with the Halting Problem. Which direction is correct: For all decidable languages L, $L \leq H$ or for all decidable languages L, $H \leq L$?

b) Assume Q is a known $\mathcal{NP}$-complete problem and Z a new decision problem in $\mathcal{NP}$. Which reduction shows that Z is $\mathcal{NP}$-complete as well? What kind of reduction is needed?

c) What is the Church-Turing Thesis?

d) Give a total binary valued function that is not computable.

2. Prove that if one $\mathcal{NP}$-complete problem lies in $\mathcal{P}$ then $\mathcal{P} = \mathcal{NP}$.

   (In your proof you can use the fact that polynomial time reducibility is transitive and, if $A <=_P B$ and $B \in \mathcal{P}$, then this implies that $A \in \mathcal{P}$ as well.)

   Hint: Start with recalling the definition of $\mathcal{NP}$-completeness.

3. Show the reduction $HP <=_P HC$. Note that this is the opposite direction than the reduction given as a sample final question.

   Hamiltonian Cycle problem (HC):
   INSTANCE: An undirected graph $G = (V, E)$.
   QUESTION: Does G have a cycle on which every vertex appears exactly once?

   Hamiltonian Path problem (HP):
   INSTANCE: An undirected graph $G = (V, E)$.
   QUESTION: Does G have a path on which every vertex appears exactly once?

   To an arbitrary instance of HP (an undirected graph) add one new vertex to obtain the corresponding instance of HC. This vertex is connected to all other vertices.

   Show that the instance of HP has an Hamiltonian Path iff the corresponding instance of HC has an Hamiltonian Cycle.

   Start by working out a small example graph.

4. Show that if both L and L' are languages in $\mathcal{NP}$, then their concatenation $L \cdot L'$ is in $\mathcal{NP}$ as well.

   Recall that $L \cdot L' := \{v \cdot w : v \in L \text{ and } w \in L'\}$.

   Hint: Assume you have non-deterministic TMs recognizing L and L', respectively. Use these two TM's to construct a non-deterministic TM for recognizing $L \cdot L'$.

5. Show that Accept(λ) is undecidable by reducing the Halting problem to it.

   Accept(λ)
   INSTANCE: An encoding $e(T)$ of a Turing machine T.
   QUESTION: Does T halt on the empty tape?

6. Show that the following function is not computable:

   $f(m, n)$ is the maximum number of steps any halting computation or crashing computation can take of a Turing machine with m states on an input of length n.

   Hint: Show that if f was computable then you could use it to decide the Halting Problem.

7. The Busy Beaver Functions is defined as follows:

For any $n \geq 0$, $f(n)$ is the largest possible number of 1's that can be left on the tape by any TM having n states and tape alphabet $\{0, 1\}$, given that the TM starts with input 1^n and eventually halts.

Complete the following proof by diagonalization that f is not computable:

Assume that there is a TM computing f. Then there is another TM T_f that computes f and has tape alphabet $\{0, 1\}$. Let $T = T_f \rightarrow T_1$, where T_1 is a TM that prints 1 in the first blank square to the right of its initial head position and halts with the tape head in its original position. Clearly, T computes the function g defined by $g(n) =$ ....

But this is impossible because ... (Complete the diagonalization argument using the number of states of T.) What does this imply about the Busy Beaver Function f?

8. In a 1742 letter to Leonhard Euler the Russian Mathematician Christian Goldbach made the following conjecture on the margin of his letter:

   Every even number is the sum of (two not necessarily distinct) primes. Goldbach considered 1 a prime.

   Examples: $2 = 1+1$, $8 = 3+5$, $10 = 5+5$, $16 = 11+5$

   This conjecture has been checked for numbers up to $4 \ 10^{14}$, but no general proof is known.

   Show that if the Halting Problem was decidable then Goldbach's Conjecture could be resolved. You can assume that you have TM that decides whether a give integer is prime.