Homework 3 Solutions, CS 132, Winter 2007

January 25, 2007

1. (10.10) *Show that if $f : \mathcal{N} \to \mathcal{N}$ is computable and strictly increasing, then the range of f (or the set of strings representing elements of the range of f) is recursive.*

   We can decide if a given y is in the range of f by computing $f(0), f(1), f(2)$, etc. until either y appears (in which case we accept) or a number larger than y does (in which case y will never appear, and we reject).

   Nondeterministic Solution (by Ben Ackerman-Hunt): Let T be the TM that computes f. T' is a two-tape NTM that decides membership in the range of f. T' works as follows:
   1. Copy input to Tape 2
   2. Delete a nondeterministically choosen number of 1's from the end of the input on Tape 1
   3. Run T on Tape 1 and halt iff T accepts

   Note: Since f is increasing we have $f(x) \geq x$.

2. (10.27) *Show that if S is uncountable and T is countable, then $S - T$ is uncountable.*

   Assume there exists a countable set T_1 and an uncountable set S_1 such that $T_1 - S_1$ is countable. We showed this two different ways in class:

   Method One: Then $T_1 \cap S_1$ is also countable, since it is a subset of T_1. Countable sets are closed under union, so $(S_1 - T_1) \cup (S_1 \cap T_1) = S_1$ is also countable, making S_1 both countable and uncountable, a contradiction.

   Method Two: $(S_1 - T_1) \cup T_1$ is countable since countable sets are closed under union, but $S_1 \subseteq (S_1 - T_1) \cup T_1$. But a countable set can't have an uncountable subset, so this is a contradiction.

3. (10.28) *Let $\mathcal{Q}$ be the set of all rational numbers or fractions, negative as well as nonnegative. Show that $\mathcal{Q}$ is countable by describing explicitly a bijection from $\mathcal{N}$ to $\mathcal{Q}$.*

   Method 1: To simplify this proof we will break $\mathcal{Q}$ into a countably infinite number of sets each denoted Q_i, where for some integer i, Q_i is the set of all rational numbers q such that $i \leq q < i + 1$. We know by theorem 10.13 that the countably infinite union of countable sets is countable; so if all Q_is are countable, then so is,

   $$Q = \bigcup_{-\infty}^{+\infty} Q_i.$$

   Now we must show all Q_is are countable. We begin by enumerating Q_0 in order by increasing denominator (omitting any fraction that is not fully reduced to avoid duplication).

   $$Q_0 = \{0, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}...\}$$

   There is a natural bijection from the ith element of Q_0 to $i \in \mathcal{N}$, so Q_0 is countable. Now for any integer i we define $Q_i = \{i + q | q \in Q_0\}$. This shows that Q is countable, but if we wish to make the bijection explicit we can do so by the same method used for theorem 10.13 (the "snake" proof): we arrange the sets vertically in the order $Q_0, Q_1, Q_{-1}, Q_2, Q_{-2}, ...$ and traverse their diagonals in the spiral pattern on page 392.

Other (Simpler) Methods: See associated handout.

4. (10.30) *Let S be the set of all infinite sequences of 0's and 1's. Show directly using a diagonal argument that S is uncountable.*

   To derive a contradiction, assume S is a countable set with elements $s_1, s_2, s_3, ...$ Notationally we will denote the jth bit of the ith sequence as $s_{i,j}$. Now consider the sequence s' such that $s'_j \neq s_{j,j}$. By assumption S contains *all* binary sequences including s'; but by definition, for any i, s' differs from the ith sequence in S on its ith bit, and so s' cannot be in S. This is a contradiction, so we can conclude that S is not be countable.

5. (10.17 extra credit) *Find a context sensitive grammer generating the language* $\{ss|s \in \{a, b\}+\}$.

   One way to get the grammer is by modifying the unrestricted grammer that is given in the book to make it context sensitive. In essence F produces strings "nondeterministically" and M duplicates them. See pg. 374 for more details

$$
\begin{array}{l}
S \rightarrow FM \\
F \rightarrow FaA \\
F \rightarrow FbB \\
Aa \rightarrow aA \\
Ab \rightarrow bA \\
Ba \rightarrow aB \\
Bb \rightarrow bB \\
AM \rightarrow Ma \\
BM \rightarrow Mb \\
F \rightarrow \Lambda \\
M \rightarrow \Lambda
\end{array}
$$

   Note that this is nearly fits the context sensitive restriction that $\beta \rightarrow \alpha$for a production $\alpha \rightarrow \beta$. The only problem is that F and M both produce 0-length strings, and so we have to replace them. Consequently we introduce F_a, F_b and M_a, M_b that both each produce the their respective subscript. This yields the following context sensetive grammer:

$$S \rightarrow F_a M_a | F_b M_b$$

$$
\begin{array}{ll}
F_a \rightarrow F_a aA | F_b aB & F_b \rightarrow F_a bA | F_b bB \\
Aa \rightarrow aA & Bb \rightarrow bB \\
Ab \rightarrow bA & Ba \rightarrow aB \\
AM_a \rightarrow M_a a & BM_a \rightarrow M_b a \\
AM_b \rightarrow M_a b & BM_b \rightarrow M_b b \\
F_a \rightarrow a & F_b \rightarrow b \\
M_a \rightarrow a & M_b \rightarrow b
\end{array}
$$