CS 132 Homework 3 Solutions

January 25, 2005

10.27 *Show that if S is uncountable and T is countable then $S - T$ is uncountable.*

We give a proof by contradiction. We begin by assuming there exists a countable set T and an uncountable set S such that $S - T$ is countable and then derive a contradiction. There are multiple ways of doing this, here are two:

Contradiction One: $S \cap T$ is also countable, since it is a subset of T and T is countable. Countable sets are closed under union, so $(S - T) \cup (S \cap T) = S$ is also countable. This makes S both countable and uncountable, a contradiction.

Contradiction Two: $(S - T) \cup T$ is countable since countable sets are closed under union, and thus $S \subseteq (S - T) \cup T$. But a countable set can't have an uncountable subset, so this too is a contradiction.

10.32 *Give an example of a set $S \subseteq 2^{\mathcal{N}}$ so that both S and $2^{\mathcal{N}} - S$ are uncountable.*

There are many such sets, one is $S = \{s \subseteq \mathcal{N} | s \text{ contains only even numbers.}\}$. We will now show that S and $2^{\mathcal{N}} - S$ are both uncountable.

Let $\mathcal{N}_e$ denote the even natural numbers. We can show that $\mathcal{N}_e$ is countably infinite by giving a bijection, $f : \mathcal{N}_e \to \mathcal{N}$; one such bijection is $f(n) = \frac{n}{2}$. Thus S, the power set of $\mathcal{N}_e$, has a bijection to $2^{\mathcal{N}}$, and is also uncountable.

Let $\mathcal{N}_o$ be the set of odd natural numbers. We can show $2^{\mathcal{N}_o}$ is uncountable with similar argument. Since $2^{\mathcal{N}_o} \subseteq 2^{\mathcal{N}} - S$, it follows that $2^{\mathcal{N}} - S$ is also uncountable.

10.33 *Show that the set of languages L over $\{0,1\}$ so that neither L nor L' is r.e. is uncountable.*

Let $\mathcal{RE} = \{L \subseteq \{0,1\}^* | L \text{ is r.e.}\}$ be the set of r.e. languages and let $\mathcal{REC} = \{\overline{\mathcal{L}} | \mathcal{L} \in \mathcal{RE}\}$ be their complements. The set in question is $2^{\{0,1\}^*} - (\mathcal{RE} \cup \mathcal{REC})$.

We know from Example 10.8 that $\mathcal{RE}$ is countable. Each language has exactly one complement, so $\mathcal{REC}$ is also countable. Since countable sets are closed under union $\mathcal{RE} \cup \mathcal{REC}$ is countable. $2^{\{0,1\}*}$ is uncountable (see Corollary 10.1).

$2^{\{0,1\}^*} - (\mathcal{RE} \cup \mathcal{REC})$ is thus an uncountable set minus a countable set, which we have shown in problem 10.27 to be uncountable.

4. *Prove with a diagonalization argument that the set $F = \{f | f : \mathcal{N} \to \{0,1\}$ is uncountably infinite.*

To derive a contradiction, we assume F is countable. If it is countable, it can be listed as $F = \{f_1, f_2, f_3, ...\}$. Now consider the function f' such that $f'(i) = 1 - f_i(i)$ for all $i \in \mathcal{N}$.

f is a function from $\mathcal{N}$ to $\{0,1\}$, so by assumption $f' \in F$; but for any particular $i \in \mathcal{N}$, $f'(i) \neq f_i(i)$ and thus $f' \neq f_i$. Consequently $f' \in F$ and $f' \notin F$, which is a contradiction; hence F is not countable.

10.25 *Show that a set S is infinite if and only if there is a bijection from S to S' for some $S' \subset S$.*

Part One: S Infinite $\Rightarrow$ There exists a bijection $f : S \rightarrow S'$

To prove this we will construct a bijection $f : S \rightarrow S'$. Since S is infinite we can partition it into two sets C and T where $C = \{c_1, c_2, \ldots\}$ is *countably* infinite. Now we can define our bijection $f : S \rightarrow S'$ where $S' = S - \{c_1\}$:

$$f(x) = \begin{cases} c_{i+1} & \text{if } x = c_i \\ x & \text{otherwise} \end{cases}$$

Part Two: S Infinite $\Leftarrow$ There exists a bijection $f : S \rightarrow S'$

Rather than show this directly, we show the contrapositive which is logically equivalent: If S is finite, then there is no bijection $f : S \rightarrow S'$.

Since S is finite, S' is also finite. Since $S' \subseteq S$, $|S'| < |S|$. Thus by the pigeon hole principle, for any $f : S \rightarrow S'$ there must exist distince elements $x_1, x_2 \in S$ such that $f(x_1) = f(x_2)$. So f is not one-to-one, and hence not a bijection.