CMPS 101
Final Review Problems

Be sure to look at the problems on all previous review sheets and homework assignments.

1. Let $G = (V, E)$ be a graph with n vertices, m edges, and k connected components.
   a. Show that if G is connected and acyclic, then $m = n - 1$. Use induction on either m or n.
   b. Show that if G is acyclic, then $m = n - k$. Use part (a).
   c. Show that if G is connected, then $m \geq n - 1$. Use induction on m.
   d. Show that in any graph G, $m \geq n - k$. Use part (c).

2. Let G be a digraph. Determine whether, at any point during a Depth First Search of G, there can exist an edge of the following kind.
   a. A tree edge that joins a white vertex to a gray vertex.
   b. A back edge that joins a black vertex to a white vertex.
   c. A forward edge that joins a gray vertex to a black vertex.
   d. A cross edge that joins a black vertex to a gray vertex.
   e. A tree edge that joins a gray vertex to a gray vertex.
   f. A forward edge that joins a black vertex to a black vertex.
   g. A cross edge that joins a white vertex to a black vertex.
   h. A back edge that joins a gray vertex to a white vertex.

3. a. State the parenthesis theorem.
   b. State the white path theorem.
   c. State the max-Heap property.
   d. State the min-Heap property.
   e. State the Binary Search Tree properties.
   f. State the Red Black Tree properties.

4. Let G be a directed graph. Prove that if G contains a directed cycle, then G contains a back edge. (Hint: use the white path theorem.)

5. Let T be a binary tree. Let $n(T)$ denote the number of nodes in T, and $h(T)$ denote the height of T. Show that $h(T) \geq \lfloor \lg(n(T)) \rfloor$.

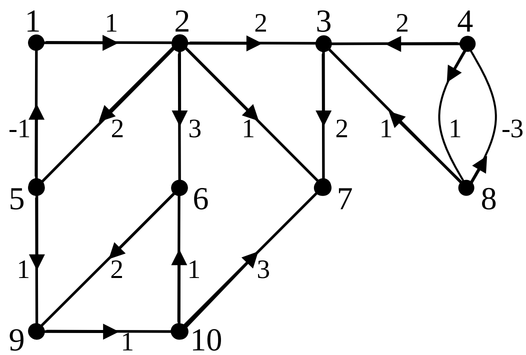
6. Re-write the algorithms Heapify, and HeapIncreaseKey from the point of view of a min-Heap, rather than a max-Heap. (In particular, HeapIncreaseKey should be renamed HeapDecreaseKey.)

7. Trace HeapSort on the following arrays. Show the state of both the array and ACBT after each swap.
   a. (9, 3, 5, 4, 8, 2, 5, 10, 12, 2, 7, 4)
   b. (5, 3, 7, 1, 10, 12, 19, 24, 5, 7, 2, 6)
   c. (9, 8, 7, 6, 5, 4, 3, 2, 1)

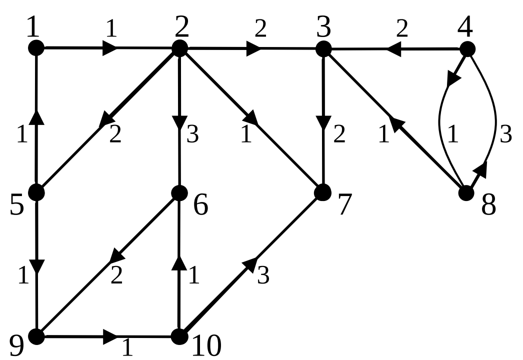
8. Let G be a directed graph, and let $s, x \in V(G)$. Suppose that after Initialize(G, s) is executed, some sequence of calls to Relax(,) results in $d[x]$ becoming finite. Show that G contains an s-x path of weight $d[x]$. (Use strong induction on the number of calls to Relax(,).)

9. Let G be a directed graph, $s, x \in V(G)$, and suppose Initialize(G, s) is executed. Show that the inequality $\delta(s, x) \leq d[x]$ is maintained over *any* sequence of calls to Relax(,). (Use the result of problem 8.)

10. Perform BellmanFord(G, s) on the weighted directed graph pictured below. Determine the d[ ] and p[ ] values for each vertex after each pass over the edge set in the main loop of BellmanFord(). Draw the resulting Shortest Paths tree, and determine the return value (true or false) of BellmanFord( ).
   a. Use $s = 2$ as source vertex.
   b. Use $s = 8$ as source vertex.



11. Change the sign on the edge weights for (5, 1) and (8, 4) in the digraph in problem 10. Perform Dijkstra(G, s) on the resulting digraph. Trace the d[ ] and p[ ] values for each vertex after each call to Relax(,), and draw the resulting Shortest Paths tree.



Note:
A question arises in this problem that did not arise in the example shown in class (see notes for 3-11-15). What shall we do when there is a tie for the smallest d-value in the min priority queue, i.e. which vertex should be extracted in such a case? If this were a programming project we would implement the priority queue as a min-heap and allow the heap operations to break ties. However this problem said nothing about creating a heap. Instead we adopt the same convention that was used in other graph algorithms to make the algorithm fully deterministic. Namely, whenever we must perform some operation on a vertex and there is more than one vertex to chose from, we take the candidate with smallest label. Therefore the first step inside Dijkstra's while loop will be: amongst those vertices with minimum d-value, extract the vertex with smallest label.
   a. Use $s = 1$ as source vertex.
   b. Use $s = 5$ as source vertex.

12. Draw the Binary Search Tree resulting from inserting the keys: 5 8 3 4 6 1 9 2 7 (in that order) into an initially empty tree. Write pseudo-code for the following recursive algorithms, and write their output when run on this tree.
   a. InOrderTreeWalk()
   b. PreOrderTreeWalk()
   c. PostOrderTreeWalk()

13. Assign colors to the nodes in the above BST in such a way that it becomes a valid RBT. Note there is more than one way to do this. Find all such color assignments.

14. Let x be a node in a Red-Black Tree, and let $N(x)$ denote the number of internal nodes in the subtree rooted at x. Show that $N(x) \geq 2^{\text{bh}(x)} - 1$. (Hint: use strong induction on $\text{height}(x)$.)

15. Let T be a Red-Black Tree having n internal nodes, and height h. Show that $h \leq 2\lg(n+1)$. (Hint: use the result of problem 14 and RBT property (4).)

16. Insert the following keys (in order) into an initially empty Binary Search Tree: 11, 2, 13, 1, 3, 12, 4, 9, 7, 10, 6, 8, 5. Draw the resulting Binary Search Tree. Prove that it is not possible to assign colors Red and Black to the nodes of this tree in such a way that the Red-Black tree properties are satisfied. (Hint: use contradiction and the result of problem 15.)

17. Insert the keys from problem 16 (in order) into an initially empty Red-Black Tree using the algorithms `RB-Insert()` and `RB-Insert-Fixup()` from lecture and the text. Draw all intermediate trees in this process.