CMPS 101
Midterm 1
Review Problems

1. Let $f(n)$ and $g(n)$ be asymptotically non-negative functions which are defined on the positive integers.
   a. State the definition of $f(n) = \mathrm{O}(g(n))$.
   b. State the definition of $f(n) = \omega(g(n))$

2. State whether the following assertions are true or false. If any statements are false, give a related statement that is true.
   a. $f(n) = O(g(n))$ implies $f(n) = o(g(n))$.
   b. $f(n) = O(g(n))$ if and only if $g(n) = \Omega(f(n))$.
   c. $f(n) = \Theta(g(n))$ if and only if $\lim_{n\to\infty}(f(n)/g(n)) = L$, where $0 < L < \infty$.

3. Prove that $\Theta(f(n)) \cdot \Theta(g(n)) = \Theta(f(n) \cdot g(n))$. In other words, if $h_1(n) = \Theta(f(n))$ and $h_2(n) = \Theta(g(n))$, then $h_1(n) \cdot h_2(n) = \Theta(f(n) \cdot g(n))$.

4. Let $f(n)$ and $g(n)$ be asymptotically positive functions (i.e. $f(n) > 0$ and $g(n) > 0$ for all sufficiently large n), and suppose $f(n) = \Theta\big(g(n)\big)$. Does it necessarily follow that $\frac{1}{f(n)} = \Theta\left(\frac{1}{g(n)}\right)$? Either prove this statement, or give a counter-example.

5. Give an example of functions $f(n)$ and $g(n)$ such that $f(n) = o(g(n))$ but $\log(f(n)) \neq o(\log(g(n)))$. (Hint: Consider $n!$ and n^n and use the corollary to Stirling's formula in the handout on common functions.)

6. Let $g(n)$ be an asymptotically non-negative function. Prove that $o\big(g(n)\big) \cap \Omega\big(g(n)\big) = \emptyset$.

7. Use limits to prove the following (these are some of the exercises at the end of the asymptotic growth rates handout):
   a. If $P(n)$ is a polynomial of degree $k \geq 0$, then $P(n) = \Theta(n^k)$.
   b. For any positive real numbers α and β: $n^\alpha = o(n^\beta)$ iff $\alpha < \beta$, $n^\alpha = \Theta(n^\beta)$ iff $\alpha = \beta$, and $n^\alpha = \omega(n^\beta)$ iff $\alpha > \beta$.
   c. For any positive real numbers a and b: $a^n = o(b^n)$ iff $a < b$, $a^n = \Theta(b^n)$ iff $a = b$, and $a^n = \omega(b^n)$ iff $a > b$.
   d. $f(n) + o\big(f(n)\big) = \Theta\big(f(n)\big)$.

8. Let $g(n) = n$ and $f(n) = n + \frac{1}{2}n^2(sin(n) + 1)$. Show that
   a. $f(n) \;=\; \Omega(g(n))$
   b. $f(n) \;\neq\; O(g(n))$
   c. $\lim_{n\to\infty}\left(\frac{f(n)}{g(n)}\right)$ does not exist, even in the sense of being infinite.

   Note: this is the 'Example C' mentioned in the handout on asymptotic growth rates.

9. Use Stirling's formula: $n! = \sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n \cdot (1 + \Theta(1/n))$, to prove that $\log(n!) = \Theta(n \log n)$.

10. Use Stirling's formula to prove that $\binom{2n}{n} = \Theta\left(\frac{4^n}{\sqrt{n}}\right)$.

11. Consider the following *sketch* of an algorithm called ProcessArray that performs some unspecified operation on a subarray $A[p \cdots r]$.

```
ProcessArray(A, p, r)      (Preconditions:  1 ≤ p and r ≤ length[A] )
Perform 1 basic operation
if p < r
     q ← ⌊(p+r)/2⌋
     ProcessArray(A, p, q)
     ProcessArray(A, q+1, r)
```

   a. Write a recurrence formula for the number $T(n)$ of basic operations performed by this algorithm when called on the full array $A[1 \cdots n]$, i.e. by ProcessArray(A, 1, n). (Hint: recall our analysis of MergeSort.)
   b. Show that the solution to this recurrence is $T(n) = 2n - 1$, whence $T(n) = \Theta(n)$.

12. Consider the following algorithm that does nothing but waste time:

```
WasteTime(n)   (pre: n ≥ 1)
if n > 1
     for i ← 1 to n^3
          waste 2 units of time
     for i ← 1 to 7
          WasteTime(⌈n/2⌉)
     waste 3 units of time
```

   a. Write a recurrence formula for the amount of time $T(n)$ wasted by this algorithm.
   b. Show that when n is an exact power of 2, the solution to this recurrence relation is given by $T(n) = 16n^3 - \frac{1}{2} - \frac{31}{2} n^{\lg 7}$, and hence $T(n) = \Theta(n^3)$.

13. Define $T(n)$ by the recurrence formula

$$T(n) = \begin{cases} 1 & 1 \le n < 3 \\ 2T(\lfloor n/3 \rfloor) + 4n & n \ge 3 \end{cases}$$

   Use Induction to show that $\forall n \ge 1$: $T(n) \le 12n$, and hence $T(n) = O(n)$.

14. Prove that all trees on n vertices have $n - 1$ edges. Do this int two ways.
   a. Induction on the number of vertices.
   b. Induction on the number of edges.

15. Define $S(n)$ for $n \in Z^+$ by the recurrence:

$$S(n) = \begin{cases} 0 & \text{if } n = 1 \\ S(\lceil n/2 \rceil) + 1 & \text{if } n \geq 2 \end{cases}$$

Use induction to prove that $S(n) \geq \lg(n)$ for all $n \geq 1$, and hence $S(n) = \Omega(\lg n)$.

16. Let $f(n)$ be a positive, increasing function that satisfies $f(n/2) = \Theta(f(n))$. Show that

$$\sum_{i=1}^{n} f(i) = \Theta(nf(n))$$

(Hint: Emulate the **Example** on page 4 of the handout on asymptotic growth rates in which it is proved that $\sum_{i=1}^{n} \boldsymbol{i^k} = \boldsymbol{\Theta}(\boldsymbol{n^{k+1}})$ for any positive integer k.)

17. Use the result of the preceding problem to give an alternate proof of $\log(n!) = \Theta(n \log(n))$ that does not use Stirling's formula.

18. Let $T(n)$ be defined by the recurrence formula

$$T(n) = \begin{cases} 1 & n = 1 \\ T(\lfloor n/2 \rfloor) + n^2 & n \geq 2 \end{cases}$$

Show that $\forall n \geq 1$: $T(n) \leq \frac{4}{3}n^2$, and hence $T(n) = O(n^2)$.

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We may not get far enough for this problem. If we do I'll let you know.

19. Define $T(n)$ by the recurrence formula:

$$T(n) = \begin{cases} 7 & 1 \leq n < 3 \\ 2T(\lfloor n/3 \rfloor) + 5 & n \geq 3 \end{cases}$$

a. Use the iteration method to determine an exact solution to the above recurrence.
b. Use the exact solution you found in part (a) to determine an asymptotic solution.
c. Use the Master Theorem to find an asymptotic solution.