CMPS 101
Midterm 1
Review Problems

1. Let $f(n)$ and $g(n)$ be asymptotically non-negative functions which are defined on the positive integers.
   a. State the definition of $f(n)=O(g(n))$.
   b. State the definition of $f(n)=\omega(g(n))$

2. State whether the following assertions are true or false. If any statements are false, give a related statement that is true.
   a. $f(n)=O(g(n))$ implies $f(n)=o(g(n))$.
   b. $f(n)=O(g(n))$ if and only if $g(n)=\Omega(f(n))$.
   c. $f(n)=\Theta(g(n))$ if and only if $\lim_{n\to\infty}(f(n)/g(n))=L$, where $0<L<\infty$.

3. Prove that $\Theta(f(n))\cdot\Theta(g(n))=\Theta(f(n)\cdot g(n))$. In other words, if $h_1(n)=\Theta(f(n))$ and $h_2(n)=\Theta(g(n))$, then $h_1(n)\cdot h_2(n)=\Theta(f(n)\cdot g(n))$.

4. Let $f(n)$ and $g(n)$ be asymptotically positive functions (i.e. $f(n)>0$ and $g(n)>0$ for all sufficiently large n), and suppose $f(n)=\Theta\big(g(n)\big)$. Does it necessarily follow that $\frac{1}{f(n)}=\Theta\left(\frac{1}{g(n)}\right)$? Either prove this statement, or give a counter-example.

5. Give an example of functions $f(n)$ and $g(n)$ such that $f(n)=o(g(n))$ but $\log(f(n))\neq o(\log(g(n)))$. (Hint: Consider $n!$ and n^n and use the corollary to Stirling's formula in the handout on common functions.)

6. Let $g(n)$ be an asymptotically non-negative function. Prove that $o((g(n))\cap\Omega(g(n))=\varnothing$.

7. Use limits to prove the following (these are some of the exercises at the end of the asymptotic growth rates handout):
   a. If $P(n)$ is a polynomial of degree $k\geq 0$, then $P(n)=\Theta(n^k)$.
   b. For any positive real numbers α and β: $n^\alpha=o(n^\beta)$ iff $\alpha<\beta$, $n^\alpha=\Theta(n^\beta)$ iff $\alpha=\beta$, and $n^\alpha=\omega(n^\beta)$ iff $\alpha>\beta$.
   c. For any positive real numbers a and b: $a^n=o(b^n)$ iff $a<b$, $a^n=\Theta(b^n)$ iff $a=b$, and $a^n=\omega(b^n)$ iff $a>b$.
   d. $f(n)+o(f(n))=\Theta(f(n))$.

8. Let $g(n) = n$ and $f(n) = n + \frac{1}{2}n^2(\sin(n)+1)$. Show that

   a. $f(n) = \Omega(g(n))$
   b. $f(n) \neq O(g(n))$
   c. $\lim_{n\to\infty}\left(\frac{f(n)}{g(n)}\right)$ does not exist, even in the sense of being infinite.

   Note: this is the 'Example C' mentioned in the handout on asymptotic growth rates.

9. Use Stirling's formula: $n! = \sqrt{2\pi n}\cdot\left(\frac{n}{e}\right)^n\cdot\left(1+\Theta(1/n)\right)$, to prove that $\log(n!) = \Theta(n\log n)$.

10. Use Stirling's formula to prove that $\binom{2n}{n} = \Theta\left(\frac{4^n}{\sqrt{n}}\right)$.

11. Consider the following *sketch* of an algorithm called ProcessArray which performs some unspecified operation on a subarray $A[p\cdots r]$.

    <u>ProcessArray(A, p, r)</u> (Preconditions: $1 \le p$ and $r \le \text{length}[A]$)
    Perform 1 basic operation
    if $p < r$
        $q \leftarrow \left\lfloor \frac{p+r}{2} \right\rfloor$
        ProcessArray(A, p, q)
        ProcessArray(A, q+1, r)

    a. Write a recurrence formula for the number $T(n)$ of basic operations performed by this algorithm when called on the full array $A[1\cdots n]$, i.e. by ProcessArray(A, 1, n). (Hint: recall our analysis of MergeSort.)
    b. Show that the solution to this recurrence is $T(n) = 2n-1$, whence $T(n) = \Theta(n)$.

12. Consider the following algorithm which does nothing but waste time:

    <u>WasteTime(n)</u> (pre: $n \ge 1$)
    if $n > 1$
        for $i \leftarrow 1$ to n^3
            waste 2 units of time
        for $i \leftarrow 1$ to 7
            WasteTime$\left(\lceil n/2 \rceil\right)$
        waste 3 units of time

    a. Write a recurrence formula which gives the amount of time $T(n)$ wasted by this algorithm.
    b. Show that when n is an exact power of 2, the solution to this recurrence relation is given by $T(n) = 16n^3 - \frac{1}{2} - \frac{31}{2}n^{\lg 7}$, and hence $T(n) = \Theta(n^3)$.

13. Define $T(n)$ by the recurrence formula

$$T(n)=\begin{cases} 1 & 1\le n<3 \\ 2T(\lfloor n/3\rfloor)\ +\ 4n & n\ge 3 \end{cases}$$

Use Induction to show that $\forall\ n\ge 1$: $T(n)\le 12n$, and hence $T(n)=O(n)$.

14. Prove that all trees on *n* vertices have $n-1$ edges. Do this int two ways.
    a. Induction on the number of vertices.
    b. Induction on the number of edges.

15. Define $S(n)$ for $n\in Z^+$ by the recurrence:

$$S(n)=\begin{cases} 0 & \text{if } n=1 \\ S(\lceil n/2\rceil)+1 & \text{if } n\ge 2 \end{cases}$$

Use induction to prove that $S(n)\ge \lg(n)$ for all $n\ge 1$, and hence $S(n)=\Omega(\lg n)$.

16. Let $f(n)$ be a positive, increasing function that satisfies $f(n/2)=\Theta(f(n))$. Show that

$$\sum_{i=1}^{n} f(i)=\Theta(nf(n))$$

(Hint: Emulate the **Example** on page 4 of the handout on asymptotic growth rates in which it is proved that $\sum_{i=1}^{n} i^k=\Theta(n^{k+1})$ for any positive integer *k*.)

17. Use the result of the preceding problem to give an alternate proof of $\log(n!)=\Theta(n\log(n))$ that does not use Stirling's formula.

18. Let $T(n)$ be defined by the recurrence formula

$$T(n)\ =\begin{cases} 1 & n=1 \\ T(\lfloor n/2\rfloor)+n^2 & n\ge 2 \end{cases}$$

Show that $\forall n\ge 1$: $T(n)\le \frac{4}{3}n^2$, and hence $T(n)=O(n^2)$.

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We may not get far enough for this problem. If we do I'll let you know.

19. Define $T(n)$ by the recurrence formula:

$$T(n)\ =\begin{cases} 7 & 1\le n<3 \\ 2T(\lfloor n/3\rfloor)+5 & n\ge 3 \end{cases}$$

a. Use the iteration method to determine a solution to the above recurrence.
b. Use the solution you found in part (a) to determine an asymptotic solution.