CMPS 101
Summer 2009
Homework Assignment 3

1. (1 Point) The last exercise in the handout entitled *Some Common Functions*.

   Use Stirling's formula to prove that $\binom{2n}{n} = \Theta\left(\frac{4^n}{\sqrt{n}}\right)$.

2. (2 Points) (Exercise 1 from the induction handout)

   Prove that for all $n \geq 1$: $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$. Do this twice:

   a. (1 Point) using form **I**Ia of the induction step.
   b. (1 Point) using form IIb of the induction step.

3. (1 Point) Exercise 2 from the induction handout)
   Define $S(n)$ for $n \in Z^+$ by the recurrence:

$$S(n) = \begin{cases} 0 & \text{if } n = 1 \\ S(\lceil n/2 \rceil) + 1 & \text{if } n \geq 2 \end{cases}$$

   Prove that $S(n) \geq \lg(n)$ for all $n \geq 1$, and hence $S(n) = \Omega(\lg n)$.

4. (2 Points)
   a. (1 Point) Let $f(n)$ be a positive, increasing function that satisfies $f(n/2) = \Theta(f(n))$. Show that $\sum_{i=1}^{n} f(i) = \Theta(nf(n))$. (Hint: follow the **Example** on page 4 of the handout on asymptotic growth rates in which it is proved that $\sum_{i=1}^{n} i^k = \Theta(n^{k+1})$ for any positive integer *k*.)
   b. (1 Point) Use the result in part (a) to deduce that $\log(n!) = \Theta(n \log(n))$.

5. (1 Point)
   Let $T(n)$ be defined by the recurrence formula

$$T(n) = \begin{cases} 1 & n = 1 \\ T(\lfloor n/2 \rfloor) + n^2 & n \geq 2 \end{cases}$$

   Show that $\forall n \geq 1$: $T(n) \leq \frac{4}{3} n^2$, and hence $T(n) = O(n^2)$. (Hint: follow Example 3 on page 3 of the handout on induction proofs.)