CMPS 201
Algorithms and Abstract Data Types

Some Common Functions

We present several common functions and estimates which occur frequently in the analysis of algorithms.

Floors and Ceilings

Given $x \in \mathbf{R}$, we denote by $\lfloor x \rfloor$ and $\lceil x \rceil$ the *floor of x* and the *ceiling of x*, respectively. These are defined to be the unique integers satisfying

$$x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1$$

Equivalently, if $x \in \mathbf{R}$ and $N \in \mathbf{Z}$ then

(1) $N = \lfloor x \rfloor$ if and only if $N \leq x < N + 1$, and
(2) $N = \lceil x \rceil$ if and only if $N - 1 < x \leq N$.

In other words:

(1) $\lfloor x \rfloor$ is the *greatest integer less than or equal to x,* and
(2) $\lceil x \rceil$ is the *least integer greater than or equal to x.*

<u>Lemma 1:</u> Let $x \in \mathbf{R}$ and $a, b \in \mathbf{Z}$. Then

(1) $a \leq x < b$ if and only if $a \leq \lfloor x \rfloor < b$, and
(2) $a < x \leq b$ if and only if $a < \lceil x \rceil \leq b$.

Proof of (1):

(i) $a \leq x$ implies $a \leq \lfloor x \rfloor$, since among all integers that are less than or equal to *x*, $\lfloor x \rfloor$ is the greatest.
(ii) $x < b$ implies $\lfloor x \rfloor < b$, since $\lfloor x \rfloor \leq x$.
(iii) $a \leq \lfloor x \rfloor$ implies $a \leq x$, since $\lfloor x \rfloor \leq x$.
(iv) $\lfloor x \rfloor < b$ implies $x < b$, since $b \leq x$ implies $b \leq \lfloor x \rfloor$, by (i). ///

Exercise: prove part (2).

<u>Lemma 2:</u> Let $x \in \mathbf{R}$ and $m \in \mathbf{Z}^+$. Then

(1) $\left\lfloor \dfrac{\lfloor x \rfloor}{m} \right\rfloor = \left\lfloor \dfrac{x}{m} \right\rfloor$, and

(2) $\left\lceil \dfrac{\lceil x \rceil}{m} \right\rceil = \left\lceil \dfrac{x}{m} \right\rceil$.

Proof of (1): Let $N = \lfloor \lfloor x \rfloor / m \rfloor$. Then

$$\begin{aligned}
& N \le \frac{\lfloor x \rfloor}{m} < N+1 \\
\Rightarrow\ & mN \le \lfloor x \rfloor < m(N+1) \\
\Rightarrow\ & mN \le x < m(N+1) \qquad \text{(by lemma 1)} \\
\Rightarrow\ & N \le x/m < N+1 \\
\Rightarrow\ & N = \lfloor x/m \rfloor,
\end{aligned}$$

and therefore $\lfloor \lfloor x \rfloor / m \rfloor = N = \lfloor x/m \rfloor$. ///

Exercise: prove part (2).

<u>Lemma 3:</u> Let $a, b, n \in \mathbf{Z}^+$. Then

(1) $\left\lfloor \dfrac{\lfloor n/a \rfloor}{b} \right\rfloor = \left\lfloor \dfrac{n}{ab} \right\rfloor$, and

(2) $\left\lceil \dfrac{\lceil n/a \rceil}{b} \right\rceil = \left\lceil \dfrac{n}{ab} \right\rceil$.

Proof: Set $x = n/a$ and $m = b$ in lemma 2. ///

Exercise

Let $n \in \mathbf{Z}$. Show that (a) $\left\lfloor \dfrac{n}{2} \right\rfloor + \left\lceil \dfrac{n}{2} \right\rceil = n$, (b) $\left\lceil \dfrac{n}{2} \right\rceil = \left\lfloor \dfrac{n+1}{2} \right\rfloor$, and (c) $\left\lfloor \dfrac{n}{2} \right\rfloor = \left\lceil \dfrac{n-1}{2} \right\rceil$.

Logarithms

Let $x, a, b \in \mathbf{R}$ where $x > 0$, $a > 1$, and $b > 1$. Then $\log_a(x)$ denotes the exponent on a which gives *x*. In other words, $\log_a(x)$ is the inverse function of a^x, which means $a^{\log_a(x)} = x$ and $\log_a(a^x) = x$. Thus

$$x = a^{\log_a(x)} = \left(b^{\log_b(a)}\right)^{\log_a(x)} = b^{\log_b(a)\cdot\log_a(x)}$$

Taking $\log_b$ of both sides of this equation yields

(*) $$\log_b(x) = \log_b(a) \cdot \log_a(x),$$

which says in particular $\log_b(x) = \text{constant} \cdot \log_a(x)$, i.e. any two log functions differ by a constant multiple. It follows that $\log_b(n) = \Theta(\log_a(n))$, so speaking in terms of asymptotic growth rates, there is really only one log function. Equation (*) implies

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$$

which shows how to convert from one log function to another. In particular $\lg(x) = \frac{\ln(x)}{\ln(2)}$. Here we use the standard notation $\lg(\,) = \log_2(\,)$, and $\ln(\,) = \log_e(\,)$, where $e = 2.71828..$. Equation (*) also implies $a^{\log_b(x)} = a^{\log_a(x)\cdot\log_b(a)} = \left(a^{\log_a(x)}\right)^{\log_b(a)} = x^{\log_b(a)}$, which gives us the useful formula

$$a^{\log_b(x)} = x^{\log_b(a)}.$$

Stirling's Formula

Let $n \in \mathbf{Z}^+$. Then $n! = \sqrt{2\pi n}\cdot\left(\frac{n}{e}\right)^n \cdot \left(1 + \Theta\left(\frac{1}{n}\right)\right)$.

Stirling's formula gives a simple way to determine asymptotic (upper, lower, and tight) bounds on functions involving $n!$. An elementary proof can be found at

http://www.sosmath.com/calculus/sequence/stirling/stirling.html

<u>Corollary:</u>

(1) $n! = o(n^n)$

(2) $n! = \omega(b^n)$ for any $b > 0$

(3) $\log(n!) = \Theta(n\log(n))$

Proof of (1):

$$\frac{n!}{n^n} = \frac{\sqrt{2\pi n}\cdot\left(\frac{n}{e}\right)^n\cdot\left(1+\Theta\left(\frac{1}{n}\right)\right)}{n^n} = \frac{\sqrt{2\pi n}\cdot\left(1+\Theta\left(\frac{1}{n}\right)\right)}{e^n} \to 0 \quad \text{as} \quad n \to \infty$$, showing that $n! = o(n^n)$. ///

Proof of (3): Taking log (any base) of both sides of Stirling's formula, we get

$$\log(n!) = \log\sqrt{2\pi n} + \log\left(\frac{n}{e}\right)^n + \log\left(1+\Theta\left(\frac{1}{n}\right)\right)$$
$$= \frac{1}{2}\log(2\pi) + \frac{1}{2}\log(n) + n\log(n) - n\log(e) + \log\left(1+\Theta\left(\frac{1}{n}\right)\right).$$

Therefore

$$\frac{\log(n!)}{n\log(n)} = 1 + \left(\text{stuff that} \to 0 \text{ as } n \to \infty\right),$$

hence $\lim_{n\to\infty}\left(\frac{\log(n!)}{n\log(n)}\right) = 1$, proving that $\log(n!) = \Theta(n\log(n))$. ///

Exercise: Prove part (2) of the corollary.

Exercise: Prove that $\binom{2n}{n} = \Theta\left(\frac{4^n}{\sqrt{n}}\right)$, where $\binom{m}{k}$ denotes the binomial coefficient $\binom{m}{k} = \frac{m!}{k!(m-k)!}$, for $0 \le k \le m$.

Exercise: Determine a number $a > 0$ such that $\binom{3n}{n} = \Theta\left(\frac{a^n}{\sqrt{n}}\right)$.