THEOREM (PARENTHESIS)
LET $u, v \in V$ AND $d[u] < d[v]$. THEN EXACTLY ONE OF THE FOLLOWING HOLDS

(1) $d[u] < f[u] < d[v] < f[v]$
$(\quad) \quad (\quad)$

OR

(2) $d[u] < d[v] < f[v] < f[u]$
$(\quad (\quad) \quad)$

REMARKS
- NECESSARILY $d[u] < f[u]$ AND $d[v] < f[v]$.
- IF $f[u] < d[v]$ THEN WE MUST BE IN CASE (1)
- IF $d[v] < f[u]$ WE WILL SHOW THAT $f[v] < f[u]$, PUTTING US IN CASE (2).
- THUS THE ESSENCE OF THE PROOF IS TO SHOW $d[u] < d[v] < f[u] < f[v]$ CANNOT OCCUR.

PROOF:
SUPPOSE $d[u] < d[v] < f[u]$. THEN v IS DISCOVERED WHILE u IS gray. $\therefore$ ALL UNDISCOVERED VERTICES ADJACENT TO v ARE VISITED, AND v IS FINISHED, BEFORE u IS FINISHED. $\therefore f[v] < f[u]$ $\therefore$ (2) HOLDS.

THUS $d[v] < f[u]$ IFF (2) HOLDS, AND $f[u] < d[v]$ IFF (1) HOLDS.

///

Rmk.
IT'S OBVIOUS FROM THE ABOVE PROOF THAT CASE (2) HOLDS IFF V IS DISCOVERED WHILE U IS GRAY, WHICH IS TRUE IFF V IS A DESCENDENT OF U IN SOME TREE OF G_p.

THEOREM (WHITE PATH)
V IS A DESCENDENT OF U IFF AT TIME d[U] THERE IS A PATH FROM U TO V CONSISTING ENTIRELY OF WHITE VERTICES.

PROOF:
($\Rightarrow$) SUPPOSE V IS A DESCENDENT OF U IN SOME TREE T OF G_p. LET W BE ANY VERTEX ALONG THE (UNIQUE) U-V PATH IN T. $\therefore$ W IS ALSO A DESCENDENT OF U, AND $\therefore$ $d[u] < d[w]$ BY ABOVE REMARK. THUS W IS WHITE AT TIME $d[u]$.

($\Leftarrow$) SUPPOSE THAT AT TIME $d[u]$ THERE EXISTS A WHITE PATH FROM U TO V, AND ASSUME TO GET A ※ THAT V IS NOT A DESCENDENT OF U.

LET V' BE THE CLOSEST VERTEX TO U ALONG THIS PATH WHICH IS NOT A DESCENDENT OF U, AND LET W BE THE PREDECESSOR OF V'. (NOTE W MAY EQUAL U.) $\therefore$ W IS A DESCENDENT OF U, AND HENCE $f[w] \le f[u]$ BY THE ABOVE REMARK.

NOTE $d[v'] < f[w]$ SINCE THERE IS AN EDGE FROM w TO v'. ALSO SINCE v' IS WHITE AT TIME $d[u]$ WE HAVE $d[u] < d[v']$. THUS

$$d[u] < d[v'] < f[w] \leq f[u],$$

AND BY THE PARENTHESIS THEOREM v' IS A DESCENDENT OF u, CONTRARY TO OUR CHOICE OF v'. $\therefore$ v IS A DESCENDENT OF u AS CLAIMED.

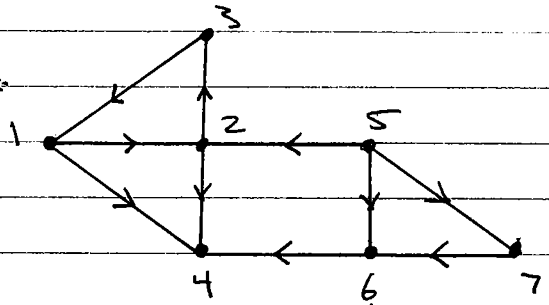
///.

CLASSIFICATION OF EDGES

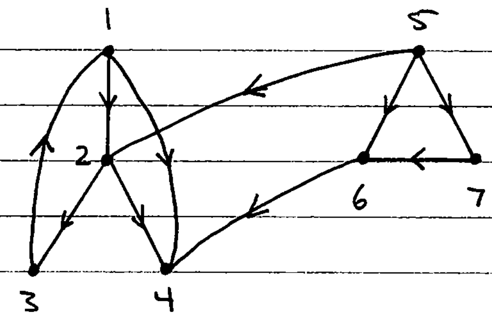
(1) TREE EDGES: BELONG TO DFS FOREST G_p.
(2) BACK EDGES: JOIN A VERTEX TO AN ANCESTOR.
(3) FORWARD EDGES: JOIN A VERTEX TO A DESCENDENT (OTHER THAN A DIRECT CHILD.)
(4) CROSS EDGES: JOIN TREE TO TREE OR COUSIN TO COUSIN.

NOTE THE DISTINCTION BETWEEN (2) AND (3) REALLY ONLY MAKES SENSE FOR DIRECTED GRAPHS. IN AN UNDIRECTED GRAPH BOTH (2) AND (3) ARE CALLED BACK EDGES.

EX.



		d	f
1 :	2 4	1	8
2 :	3 4	2	7
3 :	1	3	4
4 :		5	6
5 :	2 6 7	9	14
6 :	4	10	11
7 :	6	12	13



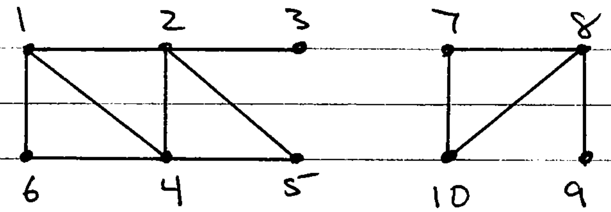
TREE: (1,2), (2,3), (2,4), (5,6), (5,7)
BACK: (3,1)
FORWARD: (1,4)
CROSS: (5,2), (6,4), (7,6)

THEOREM
IN AN UNDIRECTED GRAPH EVERY EDGE IS EITHER A TREE EDGE OR A BACK EDGE (i.e. THERE ARE NO CROSS EDGES.

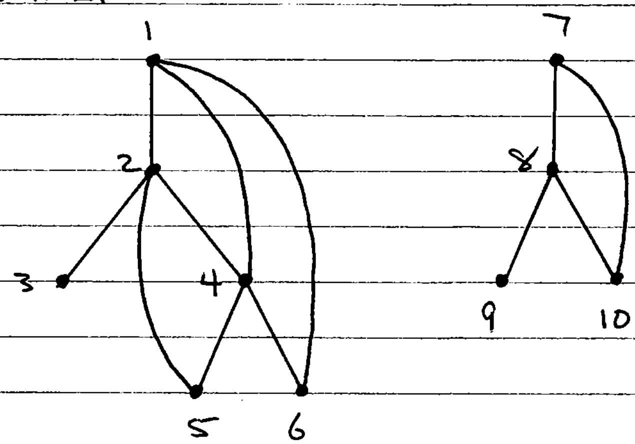
SEE P. 547 FOR THE PROOF.

COROLLARY
IN AN UNDIRECTED GRAPH EACH TREE IN A DFS FOREST SPANS A CONNECTED COMPONENT OF THE GRAPH.

Ex.



DFS FOREST:



TREE EDGES: 12, 23, 24, 45, 46, 78, 89, 810
BACK EDGES: 14, 16, 25, 710

THE DFS ALGORITHM CAN BE MODIFIED TO CLASSIFY EDGES. TO THIS END WE SAY THAT AN EDGE (u,v) IS EXPLORED IN THE DIRECTION u TO v WHEN v IS PROCESSED AS A MEMBER OF $Adj[u]$ IN LOOP 3-6 OF VISIT.

NOTE DIRECTED EDGES ARE EXPLORED ONCE WHILE UNDIRECTED EDGES ARE EXPLORED TWICE.

LEMMA

THE CLASSIFICATION OF EDGE (u,v) DEPENDS ON THE COLOR OF THE VERTEX v WHEN THE EDGE IS FIRST EXPLORED IN THE DIRECTION u TO v. IN PARTICULAR WE HAVE

(1) $\text{Color}[v] = \text{white}$ IFF (u,v) is TREE
(2) $\text{Color}[v] = \text{gray}$ IFF (u,v) is BACK
(3) $\text{Color}[v] = \text{Black}$ AND
  (3.1) $d[u] < d[v]$ IFF (u,v) is FORWARD
  (3.2) $d[v] < d[u]$ IFF (u,v) is CROSS

THE PROOF is LEFT AS AN EXERCISE, BUT SEE P. 546-548 AND PROBLEM 22.3-4.

EXERCISE

MODIFY DFS SO THAT IT PRINTS OUT THE CLASSIFICATION OF EACH EDGE AS IT IS FIRST EXPLORED. DO THIS SEPARATELY FOR:
  a.) DIRECTED GRAPHS
  b.) UNDIRECTED GRAPHS