

CMPS 101
Summer 2008
Homework Assignment 3

1. (1 Point) The last exercise in the handout entitled *Some Common Functions*.

Use Stirling's formula to prove that $\binom{2n}{n} = \Theta\left(\frac{4^n}{\sqrt{n}}\right)$.

2. (2 Points) (Exercise 1 from the induction handout)

Prove that for all $n \geq 1$: $\sum_{i=1}^n i^3 = \left(\frac{n(n+1)}{2}\right)^2$. Do this twice:

- (1 Point) using form IIa of the induction step.
 - (1 Point) using form IIb of the induction step.
3. (1 Point) Exercise 2 from the induction handout)
Define $S(n)$ for $n \in \mathbb{Z}^+$ by the recurrence:

$$S(n) = \begin{cases} 0 & \text{if } n = 1 \\ S(\lceil n/2 \rceil) + 1 & \text{if } n \geq 2 \end{cases}$$

Prove that $S(n) \geq \lg(n)$ for all $n \geq 1$, and hence $S(n) = \Omega(\lg n)$.

4. (10 Points)

Let $f(n)$ be a positive, increasing function that satisfies $f(n/2) = \Theta(f(n))$. Show that

$$\sum_{i=1}^n f(i) = \Theta(nf(n))$$

(Hint: follow the example on page 4 of the handout on asymptotic growth rates.)

5. (10 Points)

Let $T(n)$ be defined by the recurrence formula

$$T(n) = \begin{cases} 1 & n = 1 \\ T(\lfloor n/2 \rfloor) + n^2 & n \geq 2 \end{cases}$$

Show that $\forall n \geq 1$: $T(n) \leq \frac{4}{3}n^2$, and hence $T(n) = O(n^2)$. (Hint: follow example 3 on page 3 of the handout on induction proofs.)