

## Asymptotic Growth of Functions

We introduce several types of asymptotic notation which are used to compare the relative performance and efficiency of algorithms. As we've seen in comparing InsertionSort and MergeSort, the asymptotic growth rate of an algorithm gives a simple, and machine independent, characterization of the algorithm's complexity.

**Definition** Let  $g(n)$  be a function. The set  $O(g(n))$  is defined as

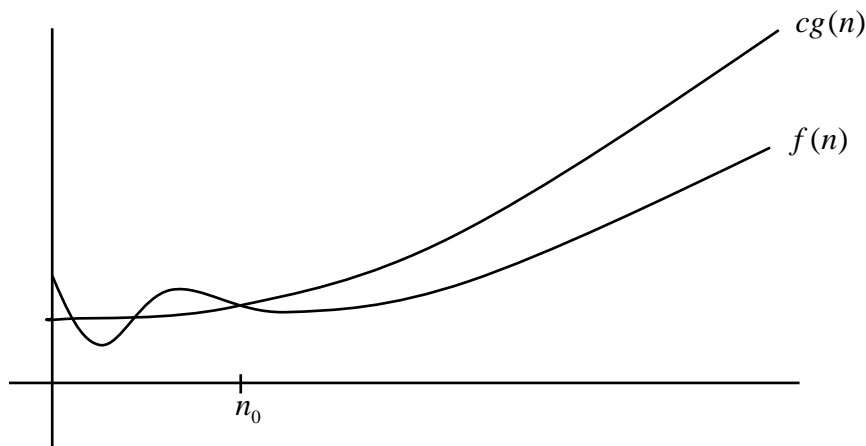
$$O(g(n)) = \{ f(n) \mid \exists c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq f(n) \leq cg(n) \}.$$

In other words,  $f(n) \in O(g(n))$  if and only if there exist positive constants  $c$ , and  $n_0$ , such that for all  $n \geq n_0$ , the inequality  $0 \leq f(n) \leq cg(n)$  is satisfied. We say that  $f(n)$  is *Big O* of  $g(n)$ , or that  $g(n)$  is an *asymptotic upper bound* for  $f(n)$ .

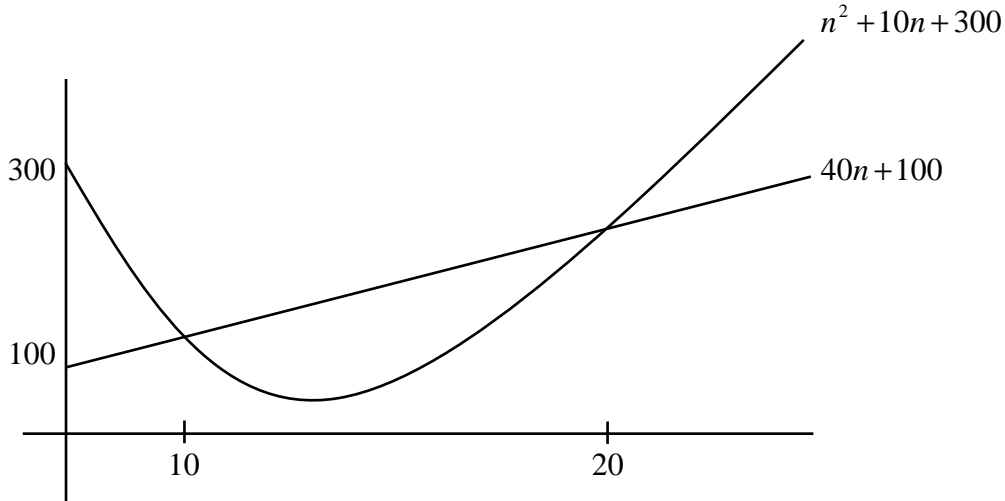
We often abuse notation slightly by writing  $f(n) = O(g(n))$  to mean  $f(n) \in O(g(n))$ . Actually  $f(n) \in O(g(n))$  is also an abuse of notation. We should really write  $f \in O(g)$  since we have defined a set of functions, not a set of numbers. The notational convention  $O(g(n))$  is useful since it allows us to refer to the set  $O(n^3)$  say, without having to introduce a function symbol for the polynomial  $n^3$ . Observe that if  $f(n) = O(g(n))$  then  $f(n)$  is *asymptotically non-negative*, i.e.  $f(n)$  is non-negative for all sufficiently large  $n$ , and likewise for  $g(n)$ . We make the blanket assumption from now on that all functions under discussion are asymptotically non-negative.

In practice we will be concerned with integer valued functions of a (positive) integer  $n$  ( $g : \mathbf{Z}^+ \rightarrow \mathbf{Z}^+$ ). However, in what follows, it is useful to consider  $n$  to be a continuous real variable taking positive values and  $g$  to be real valued function ( $g : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ ).

Geometrically  $f(n) = O(g(n))$  says:



**Example**  $40n+100 = O(n^2+10n+300)$ . Observe that  $0 \leq 40n+100 \leq n^2+10n+300$  for all  $n \geq 20$ . Thus we may take  $n_0 = 20$  and  $c = 1$  in the definition.

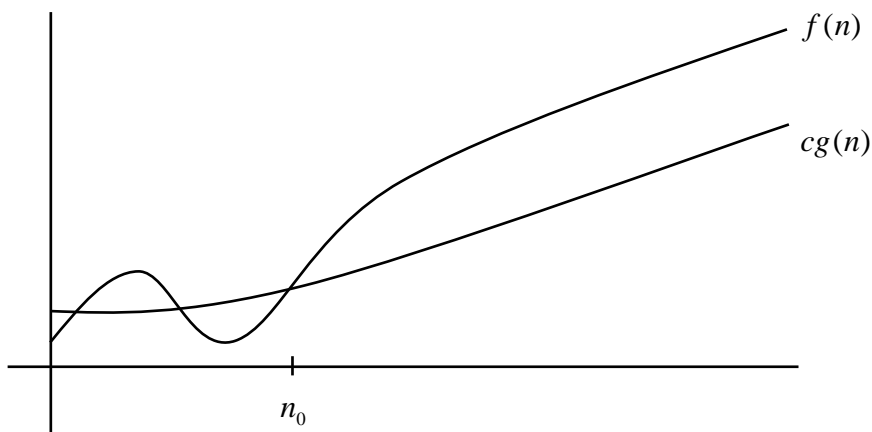


In fact  $an+b = O(cn^2+dn+e)$  for any constants  $a-e$ , and more generally  $p(n) = O(q(n))$  whenever  $p(n)$  and  $q(n)$  are polynomials satisfying  $\deg(p) \leq \deg(q)$  as we shall see.

**Definition** Let  $g(n)$  be a function and define the set  $\Omega(g(n))$  to be

$$\Omega(g(n)) = \{ f(n) \mid \exists c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq cg(n) \leq f(n) \}.$$

We say  $f(n)$  is *big Omega* of  $g(n)$ , and that  $g(n)$  is an *asymptotic lower bound* for  $f(n)$ . As before we write  $f(n) = \Omega(g(n))$  to mean  $f(n) \in \Omega(g(n))$ . The geometric interpretation is:



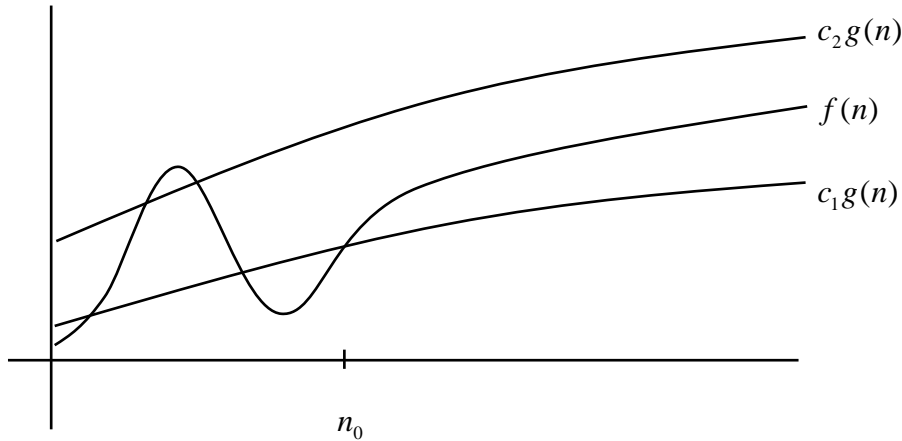
**Example** Prove that  $f(n) = O(g(n))$  if and only if  $g(n) = \Omega(f(n))$ .

**Proof:** If  $f(n) = O(g(n))$  then there exist positive numbers  $c_1, n_1$  such that  $0 \leq f(n) \leq c_1g(n)$  for all  $n \geq n_1$ . Let  $c_2 = 1/c_1$  and  $n_2 = n_1$ . Then  $0 \leq c_2f(n) \leq g(n)$  for all  $n \geq n_2$ , proving  $g(n) = \Omega(f(n))$ . The converse is similar and we leave it to the reader. // //

**Definition** Let  $g(n)$  be a function and define the set  $\Theta(g(n)) = O(g(n)) \cap \Omega(g(n))$ . Equivalently

$$\Theta(g(n)) = \{ f(n) \mid \exists c_1 > 0, \exists c_2 > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n) \}.$$

We write  $f(n) = \Theta(g(n))$  and say the  $g(n)$  is an *asymptotically tight bound* for  $f(n)$ , or that  $f(n)$  and  $g(n)$  are *asymptotically equivalent*. We interpret this geometrically as:



**Exercise** Prove that  $f(n) = \Theta(g(n))$  if and only if  $g(n) = \Theta(f(n))$ .

**Exercise** Let  $g(n)$  be any function, and let  $c > 0$ . Prove that  $cg(n) = O(g(n))$ , and  $cg(n) = \Omega(g(n))$ , whence  $cg(n) = \Theta(g(n))$ .

**Example** Prove that  $\sqrt{n+10} = \Theta(\sqrt{n})$ .

**Proof:** According to the definition, we must find positive numbers  $c_1, c_2, n_0$ , such that the inequality  $0 \leq c_1 \sqrt{n} \leq \sqrt{n+10} \leq c_2 \sqrt{n}$  holds for all  $n \geq n_0$ . Pick  $c_1 = 1$ ,  $c_2 = \sqrt{2}$ , and  $n_0 = 10$ . Then if  $n \geq n_0$  we have:

$$\begin{aligned} & -10 \leq 0 \quad \text{and} \quad 10 \leq n \\ \therefore & -10 \leq (1-1)n \quad \text{and} \quad 10 \leq (2-1)n \\ \therefore & -10 \leq (1-c_1^2)n \quad \text{and} \quad 10 \leq (c_2^2-1)n \\ \therefore & c_1^2 n \leq n+10 \quad \text{and} \quad n+10 \leq c_2^2 n, \\ \therefore & c_1^2 n \leq n+10 \leq c_2^2 n, \\ \therefore & c_1 \sqrt{n} \leq \sqrt{n+10} \leq c_2 \sqrt{n}, \end{aligned}$$

as required. ///

The reader may find our choice of values for the constants  $c_1, c_2, n_0$  somewhat mysterious. Adequate values for these constants can usually be obtained by working backwards algebraically from the inequality to be proved. Notice that in this example there are many valid choices. For instance one checks easily that  $c_1 = \sqrt{1/2}$ ,  $c_2 = \sqrt{3/2}$ , and  $n_0 = 20$  work equally well.

**Exercise** Let  $a, b$  be real numbers with  $b > 0$ . Prove directly from the definition (as above) that  $(n+a)^b = \Theta(n^b)$ . (In what follows we learn a much easier way to prove this.)

**Example** Prove that if  $h(n) = O(g(n))$  and  $f(n) \leq h(n)$  for all sufficiently large  $n$ , then  $f(n) = O(g(n))$ .

**Proof:** The above hypotheses say that there exist positive numbers  $c$  and  $n_1$  such that  $h(n) \leq cg(n)$  for all  $n \geq n_1$ . Also there exists  $n_2$  such that  $0 \leq f(n) \leq h(n)$  for all  $n \geq n_2$ . (Recall  $f(n)$  is assumed to be asymptotically non-negative.) Then for all  $n \geq n_0 = \max(n_1, n_2)$  we have  $0 \leq f(n) \leq cg(n)$ , showing that  $f(n) = O(g(n))$ . ///

**Exercise** Prove that if  $h_1(n) \leq f(n) \leq h_2(n)$  for all sufficiently large  $n$ , where  $h_1(n) = \Omega(g(n))$  and  $h_2(n) = O(g(n))$ , then  $f(n) = \Theta(g(n))$ .

**Example** Let  $k \geq 1$  be a fixed integer. Prove that  $\sum_{i=1}^n i^k = \Theta(n^{k+1})$ .

**Proof:** Observe that  $\sum_{i=1}^n i^k \leq \sum_{i=1}^n n^k = n \cdot n^k = n^{k+1} = O(n^{k+1})$ , and

$$\sum_{i=1}^n i^k \geq \sum_{i=\lceil n/2 \rceil}^n i^k \geq \sum_{i=\lceil n/2 \rceil}^n (n/2)^k \geq \lceil n/2 \rceil \cdot (n/2)^k \geq (n/2)(n/2)^k = (1/2)^{k+1} n^{k+1} = \Omega(n^{k+1}).$$

By the result of the preceding exercise we may conclude  $\sum_{i=1}^n i^k = \Theta(n^{k+1})$ . ///

When asymptotic notation appears in a formula such as  $T(n) = 2T(n/2) + \Theta(n)$  we interpret  $\Theta(n)$  to stand for some anonymous function in the class  $\Theta(n)$ . For example  $3n^3 + 4n^2 - 2n + 1 = 3n^3 + \Theta(n^2)$ . Here  $\Theta(n^2)$  stands for  $4n^2 - 2n + 1$ , which belongs to the class  $\Theta(n^2)$ . The expression  $\sum_{i=1}^n \Theta(i)$  can be puzzling. On the surface it stands for  $\Theta(1) + \Theta(2) + \Theta(3) + \dots + \Theta(n)$  which is meaningless since  $\Theta(\text{constant})$  consists of all functions which are bounded above by some constant. We interpret  $\Theta(i)$  in this expression to stand for a *single* function  $f(i)$  in the class  $\Theta(i)$ , evaluated at  $i = 1, 2, 3, \dots, n$ .

**Exercise** Prove that  $\sum_{i=1}^n \Theta(i) = \Theta(n^2)$ . The left hand side stands for a single function  $f(i)$  summed for  $i = 1, 2, 3, \dots, n$ . By the previous exercise it is sufficient to show that  $h_1(n) \leq \sum_{i=1}^n f(i) \leq h_2(n)$  for all sufficiently large  $n$ , where  $h_1(n) = \Omega(n^2)$  and  $h_2(n) = O(n^2)$ .

**Definition**  $o(g(n)) = \{ f(n) \mid \forall c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq f(n) < cg(n) \}$ . We say that  $g(n)$  is a *strict Asymptotic upper bound* for  $f(n)$  and write  $f(n) = o(g(n))$  as before.

**Lemma**  $f(n) = o(g(n))$  if and only if  $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 0$ .

**Proof:** Observe that  $f(n) = o(g(n))$  if and only if  $\forall c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq \frac{f(n)}{g(n)} < c$ , which is the very definition of the limit statement  $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 0$ . ///

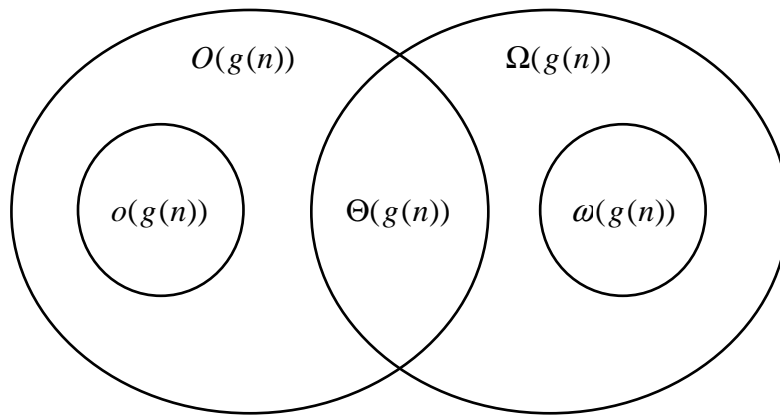
**Example**  $\lg(n) = o(n)$  since  $\lim_{n \rightarrow \infty} \frac{\lg(n)}{n} = 0$ . (Apply l'Hopitals rule.)

**Example**  $n^k = o(b^n)$  for any  $k > 0$  and  $b > 1$  since  $\lim_{n \rightarrow \infty} \frac{n^k}{b^n} = 0$ . (Apply l'Hopitals rule  $\lceil k \rceil$  times.) In other words, any polynomial grows strictly slower than any exponential.

By comparing definitions of  $o(g(n))$  and  $O(g(n))$  one sees immediately that  $o(g(n)) \subseteq O(g(n))$ . Also no function can belong to both  $o(g(n))$  and  $\Omega(g(n))$ , as is easily verified (exercise), so that  $o(g(n)) \cap \Omega(g(n)) = \emptyset$ . Thus  $o(g(n)) \subseteq O(g(n)) - \Theta(g(n))$ .

**Definition**  $\omega(g(n)) = \{ f(n) \mid \forall c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq cg(n) < f(n) \}$ . Here we say that  $g(n)$  is a *strict asymptotic lower bound* for  $f(n)$  and write  $f(n) = \omega(g(n))$ .

**Exercise** Prove that  $f(n) = \omega(g(n))$  if and only if  $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = \infty$ . Also prove  $\omega(g(n)) \cap O(g(n)) = \emptyset$ , whence  $\omega(g(n)) \subseteq \Omega(g(n)) - \Theta(g(n))$ . The following picture emerges.



**Lemma** If  $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = L$ , where  $0 \leq L < \infty$ , then  $f(n) = O(g(n))$ .

**Proof:** The limit statement says  $\forall \varepsilon > 0, \exists n_0 > 0, \forall n \geq n_0 : \left| \frac{f(n)}{g(n)} - L \right| < \varepsilon$ . Since this holds for all  $\varepsilon$ , we may set  $\varepsilon = 1$ . Then for all  $n \geq n_0$ :

$$\begin{aligned} & \left| \frac{f(n)}{g(n)} - L \right| < 1 \\ \therefore & -1 < \frac{f(n)}{g(n)} - L < 1 \end{aligned}$$

$$\begin{aligned} \therefore \quad & \frac{f(n)}{g(n)} < L+1 \\ \therefore \quad & f(n) < (L+1) \cdot g(n). \end{aligned}$$

Now taking  $c = L+1$  in the definition of  $O$  yields  $f(n) = O(g(n))$  as claimed. ///

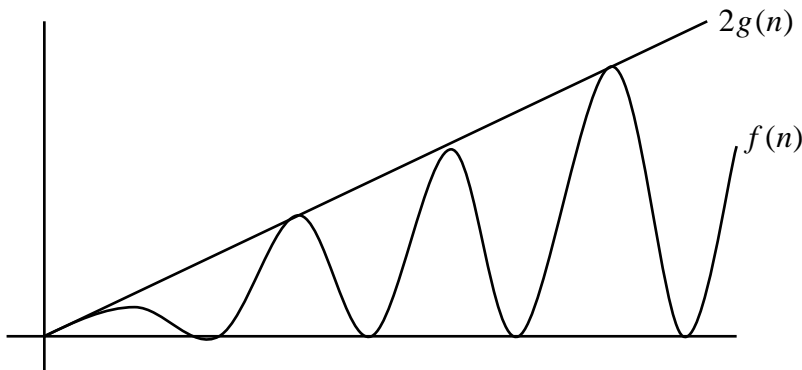
**Lemma** If  $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = L$ , where  $0 < L \leq \infty$ , then  $f(n) = \Omega(g(n))$ .

**Proof:** The limit statement implies  $\lim_{n \rightarrow \infty} \frac{g(n)}{f(n)} = L'$ , where  $L' = 1/L$  and hence  $0 \leq L' < \infty$ . By the previous lemma  $g(n) = O(f(n))$ , and therefore  $f(n) = \Omega(g(n))$ . ///

**Exercise** Prove that if  $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = L$ , where  $0 < L < \infty$ , then  $f(n) = \Theta(g(n))$ .

Although  $o(g(n))$ ,  $\omega(g(n))$ , and a certain subset of  $\Theta(g(n))$  are characterized by limits, the full sets  $O(g(n))$ ,  $\Omega(g(n))$ , and  $\Theta(g(n))$  have no such characterization as the following examples show.

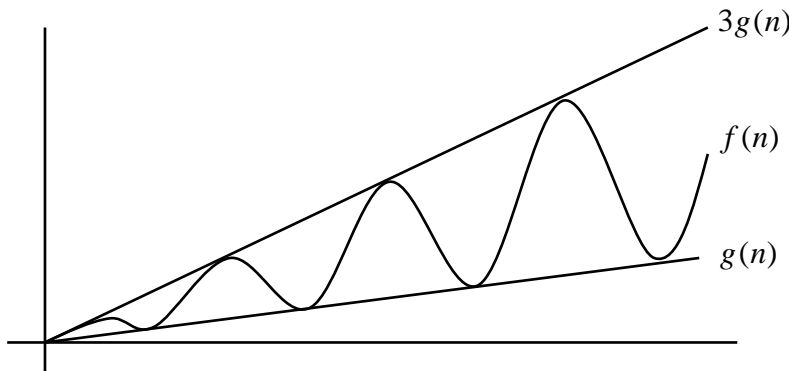
**Example A** Let  $g(n) = n$  and  $f(n) = (1 + \sin(n)) \cdot n$ .



Clearly  $f(n) = O(g(n))$ , but  $\frac{f(n)}{g(n)} = 1 + \sin(n)$ , whose limit does not exist, whence  $f(n) \neq o(g(n))$ .

Observe also that  $f(n) \neq \Omega(g(n))$  (why?). Therefore  $f(n) \in O(g(n)) - \Theta(g(n)) - o(g(n))$ , showing that the containment  $o(g(n)) \subseteq O(g(n)) - \Theta(g(n))$  is strict in general.

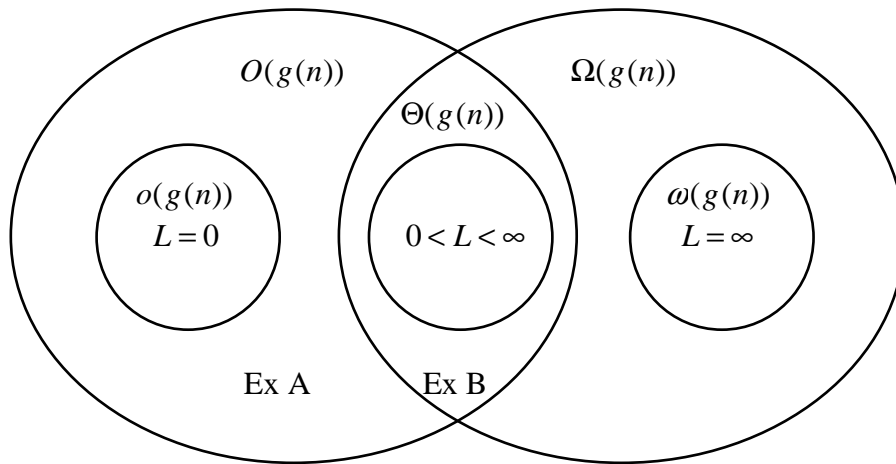
**Example B** Let  $g(n) = n$  and  $f(n) = (2 + \sin(n)) \cdot n$ .



Since  $n \leq (2 + \sin(n)) \cdot n \leq 3n$  for all  $n \geq 0$ , we have  $f(n) = \Theta(g(n))$ , but  $\frac{f(n)}{g(n)} = 2 + \sin(n)$  whose limit does not exist.

**Exercise** Find functions  $f(n)$  and  $g(n)$  such that  $f(n) \in \Omega(g(n)) - \Theta(g(n))$ , but  $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)}$  does not exist (even in the sense of being infinite), so that  $f(n) \neq \omega(g(n))$ .

The preceding limit theorems and counter-examples can be summarized in the following diagram. Here  $L$  denotes the limit  $L = \lim_{n \rightarrow \infty} \frac{f(n)}{g(n)}$ , if it exists.



In spite of the above counter-examples, the preceding limit theorems are a very useful tool for establishing asymptotic comparisons between functions. For instance recall the earlier exercise to show  $(n+a)^b = \Theta(n^b)$  for real numbers  $a$ , and  $b$  with  $b > 0$ . The result follows immediately from

$$\lim_{n \rightarrow \infty} \frac{(n+a)^b}{n^b} = \lim_{n \rightarrow \infty} \left(1 + \frac{a}{n}\right)^b = 1^b = 1,$$

since  $0 < 1 < \infty$ .

**Exercise** Use limits to prove the following:

- $n \lg(n) = o(n^2)$  (here  $\lg(n)$  denotes the base 2 logarithm of  $n$ .)
- $n^5 2^n = \omega(n^{10})$ .
- If  $P(n)$  is a polynomial of degree  $k \geq 0$ , then  $P(n) = \Theta(n^k)$ .
- For any positive real numbers  $\alpha$  and  $\beta$ :  $n^\alpha = o(n^\beta)$  iff  $\alpha < \beta$ ,  $n^\alpha = \Theta(n^\beta)$  iff  $\alpha = \beta$ , and  $n^\alpha = \omega(n^\beta)$  iff  $\alpha > \beta$ .
- For any positive real numbers  $a$  and  $b$ :  $a^n = o(b^n)$  iff  $a < b$ ,  $a^n = \Theta(b^n)$  iff  $a = b$ , and  $a^n = \omega(b^n)$  iff  $a > b$ .
- For any positive real numbers  $a$  and  $b$ :  $\log_a(n) = \Theta(\log_b(n))$ .
- $f(n) + o(f(n)) = \Theta(f(n))$ .

There is an analogy between the asymptotic comparison of functions  $f(n)$  and  $g(n)$ , and the comparison of real numbers  $x$  and  $y$ .

$$\begin{aligned}f(n) = O(g(n)) &\sim x \leq y \\f(n) = \Theta(g(n)) &\sim x = y \\f(n) = \Omega(g(n)) &\sim x \geq y \\f(n) = o(g(n)) &\sim x < y \\f(n) = \omega(g(n)) &\sim x > y\end{aligned}$$

If both  $f$  and  $g$  are polynomials of degrees  $x$  and  $y$  respectively, then the analogy is exact, as can be seen from parts (c) and (d) of the preceding exercise. In general though, the analogy is *not* exact since there exist pairs of functions which are not comparable, whereas any two real numbers are comparable.

**Exercise** Let  $f(n) = n^{\sin(n)}$  and  $g(n) = \sqrt{n}$ . Show that  $f(n)$  and  $g(n)$  are incomparable, i.e.  $f(n)$  is in neither of the classes  $O(g(n))$  nor  $\Omega(g(n))$ .