

CMPS 101
Midterm 1
Review Problems

- Let $f(n)$ and $g(n)$ be asymptotically non-negative functions which are defined on the positive integers.
 - State the definition of $f(n) = O(g(n))$.
 - State the definition of $f(n) = \omega(g(n))$
- State whether the following assertions are true or false. If any statements are false, give a related statement that is true.
 - $f(n) = O(g(n))$ implies $f(n) = o(g(n))$.
 - $f(n) = O(g(n))$ if and only if $g(n) = \Omega(f(n))$.
 - $f(n) = \Theta(g(n))$ if and only if $\lim_{n \rightarrow \infty} (f(n)/g(n)) = L$, where $0 < L < \infty$.
- Prove that $\Theta(f(n)) \cdot \Theta(g(n)) = \Theta(f(n) \cdot g(n))$. In other words, if $h_1(n) = \Theta(f(n))$ and $h_2(n) = \Theta(g(n))$, then $h_1(n) \cdot h_2(n) = \Theta(f(n) \cdot g(n))$.
- Let $f(n)$ and $g(n)$ be asymptotically positive functions (i.e. $f(n) > 0$ and $g(n) > 0$ for all sufficiently large n), and suppose $f(n) = \Theta(g(n))$. Does it necessarily follow that $\frac{1}{f(n)} = \Theta\left(\frac{1}{g(n)}\right)$? Either prove this statement, or give a counter-example.
- Give an example of functions $f(n)$ and $g(n)$ such that $f(n) = o(g(n))$ but $\log(f(n)) \neq o(\log(g(n)))$. (Hint: Consider $n!$ and n^n and use the corollary to Stirling's formula in the handout on common functions.)
- Let $g(n)$ be an asymptotically non-negative function. Prove that $o(g(n)) \cap \Omega(g(n)) = \emptyset$.
- Use limits to prove the following (these are some of the exercises at the end of the asymptotic growth rates handout):
 - If $P(n)$ is a polynomial of degree $k \geq 0$, then $P(n) = \Theta(n^k)$.
 - For any positive real numbers α and β : $n^\alpha = o(n^\beta)$ iff $\alpha < \beta$, $n^\alpha = \Theta(n^\beta)$ iff $\alpha = \beta$, and $n^\alpha = \omega(n^\beta)$ iff $\alpha > \beta$.
 - For any positive real numbers a and b : $a^n = o(b^n)$ iff $a < b$, $a^n = \Theta(b^n)$ iff $a = b$, and $a^n = \omega(b^n)$ iff $a > b$.
 - $f(n) + o(f(n)) = \Theta(f(n))$.
- Let $g(n) = n$ and $f(n) = n + \frac{1}{2}n^2(\sin(n) + 1)$. Show that
 - $f(n) = \Omega(g(n))$
 - $f(n) \neq O(g(n))$
 - $\lim_{n \rightarrow \infty} \left(\frac{f(n)}{g(n)}\right)$ does not exist, even in the sense of being infinite.Note: this is the 'Example C' mentioned in the handout on asymptotic growth rates.

9. Use Stirling's formula: $n! = \sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n \cdot (1 + \Theta(1/n))$, to prove that $\log(n!) = \Theta(n \log n)$.
10. Use Stirling's formula to prove that $\binom{2n}{n} = \Theta\left(\frac{4^n}{\sqrt{n}}\right)$.
11. Consider the following *sketch* of an algorithm called `ProcessArray` which performs some unspecified operation on a subarray $A[p \dots r]$.

ProcessArray(A, p, r) (Preconditions: $1 \leq p$ and $r \leq \text{length}[A]$)

1. Perform 1 basic operation
2. if $p < r$
3. $q \leftarrow \lfloor \frac{p+r}{2} \rfloor$
4. ProcessArray(A, p, q)
5. ProcessArray(A, q+1, r)

- a. Write a recurrence formula for the number $T(n)$ of basic operations performed by this algorithm when called on the full array $A[1 \dots n]$, i.e. by `ProcessArray(A, 1, n)`. (Hint: recall our analysis of MergeSort.)
 - b. Show that the solution to this recurrence is $T(n) = 2n - 1$, whence $T(n) = \Theta(n)$.
12. Consider the following algorithm which does nothing but waste time:

WasteTime(n) (pre: $n \geq 1$)

1. if $n > 1$
2. for $i \leftarrow 1$ to n^3
3. waste 2 units of time
4. for $i \leftarrow 1$ to 7
5. WasteTime($\lfloor n/2 \rfloor$)
6. waste 3 units of time

- a. Write a recurrence formula which gives the amount of time $T(n)$ wasted by this algorithm.
 - b. Show that when n is an exact power of 2, the solution to this recurrence relation is given by $T(n) = 16n^3 - \frac{1}{2} - \frac{31}{2}n^{\lg 7}$, and hence $T(n) = \Theta(n^3)$.
13. Define $T(n)$ by the recurrence formula

$$T(n) = \begin{cases} 1 & 1 \leq n < 3 \\ 2T(\lfloor n/3 \rfloor) + 4n & n \geq 3 \end{cases}$$

Use Induction to show that $\forall n \geq 1: T(n) \leq 12n$, and hence $T(n) = O(n)$.

14. Prove that all trees on n vertices have $n - 1$ edges. Do this in two ways.
- a. Induction on the number of vertices.
 - b. Induction on the number of edges.

15. Define $S(n)$ for $n \in \mathbb{Z}^+$ by the recurrence:

$$S(n) = \begin{cases} 0 & \text{if } n = 1 \\ S(\lfloor n/2 \rfloor) + 1 & \text{if } n \geq 2 \end{cases}$$

Use induction to prove that $S(n) \geq \lg(n)$ for all $n \geq 1$, and hence $S(n) = \Omega(\lg n)$.

16. Let $f(n)$ be a positive, increasing function that satisfies $f(n/2) = \Theta(f(n))$. Show that

$$\sum_{i=1}^n f(i) = \Theta(nf(n))$$

(Hint: Emulate the **Example** on page 4 of the handout on asymptotic growth rates in which it is proved that $\sum_{i=1}^n i^k = \Theta(n^{k+1})$ for any positive integer k .)

17. Use the result of the preceding problem to give an alternate proof of $\log(n!) = \Theta(n \log(n))$ that does not use Stirling's formula.

18. Let $T(n)$ be defined by the recurrence formula

$$T(n) = \begin{cases} 1 & n = 1 \\ T(\lfloor n/2 \rfloor) + n^2 & n \geq 2 \end{cases}$$

Show that $\forall n \geq 1: T(n) \leq \frac{4}{3}n^2$, and hence $T(n) = O(n^2)$.

We may not get far enough for this problem. If we do I'll let you know.

19. Define $T(n)$ by the recurrence formula:

$$T(n) = \begin{cases} 7 & 1 \leq n < 3 \\ 2T(\lfloor n/3 \rfloor) + 5 & n \geq 3 \end{cases}$$

- Use the iteration method to determine an exact solution to the above recurrence.
- Use the exact solution you found in part (a) to determine an asymptotic solution.
- Use the Master Theorem to find an asymptotic solution.