

CMPS 201
Algorithms and Abstract Data Types

Some Common Functions

We present several common functions and estimates which occur frequently in the analysis of algorithms.

Floors and Ceilings

Given $x \in \mathbf{R}$, we denote by $\lfloor x \rfloor$ and $\lceil x \rceil$ the *floor of x* and the *ceiling of x* , respectively. These are defined to be the unique integers satisfying

$$x-1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x+1$$

Equivalently, if $x \in \mathbf{R}$ and $N \in \mathbf{Z}$ then

- (1) $N = \lfloor x \rfloor$ if and only if $N \leq x < N+1$, and
- (2) $N = \lceil x \rceil$ if and only if $N-1 < x \leq N$.

In other words:

- (1) $\lfloor x \rfloor$ is the *greatest integer less than or equal to x* , and
- (2) $\lceil x \rceil$ is the *least integer greater than or equal to x* .

Lemma 1: Let $x \in \mathbf{R}$ and $a, b \in \mathbf{Z}$. Then

- (1) $a \leq x < b$ if and only if $a \leq \lfloor x \rfloor < b$, and
- (2) $a < x \leq b$ if and only if $a < \lceil x \rceil \leq b$.

Proof of (1):

- (i) $a \leq x$ implies $a \leq \lfloor x \rfloor$, since among all integers that are less than or equal to x , $\lfloor x \rfloor$ is the greatest.
- (ii) $x < b$ implies $\lfloor x \rfloor < b$, since $\lfloor x \rfloor \leq x$.
- (iii) $a \leq \lfloor x \rfloor$ implies $a \leq x$, since $\lfloor x \rfloor \leq x$.
- (iv) $\lfloor x \rfloor < b$ implies $x < b$, since $b \leq x$ implies $b \leq \lfloor x \rfloor$, by (i). ///

Exercise: prove part (2).

Lemma 2: Let $x \in \mathbf{R}$ and $m \in \mathbf{Z}^+$. Then

- (1) $\left\lfloor \frac{\lfloor x \rfloor}{m} \right\rfloor = \left\lfloor \frac{x}{m} \right\rfloor$, and
- (2) $\left\lceil \frac{\lceil x \rceil}{m} \right\rceil = \left\lceil \frac{x}{m} \right\rceil$.

Proof of (1): Let $N = \lfloor \lfloor x \rfloor / m \rfloor$. Then

$$\begin{aligned} N &\leq \frac{\lfloor x \rfloor}{m} < N + 1 \\ \Rightarrow mN &\leq \lfloor x \rfloor < m(N + 1) \\ \Rightarrow mN &\leq x < m(N + 1) && \text{(by lemma 1)} \\ \Rightarrow N &\leq x/m < N + 1 \\ \Rightarrow N &= \lfloor x/m \rfloor, \end{aligned}$$

and therefore $\lfloor \lfloor x \rfloor / m \rfloor = N = \lfloor x/m \rfloor$. ///

Exercise: prove part (2).

Lemma 3: Let $a, b, n \in \mathbf{Z}^+$. Then

$$\begin{aligned} (1) \quad \left\lfloor \frac{\lfloor n/a \rfloor}{b} \right\rfloor &= \left\lfloor \frac{n}{ab} \right\rfloor, \text{ and} \\ (2) \quad \left\lceil \frac{\lceil n/a \rceil}{b} \right\rceil &= \left\lceil \frac{n}{ab} \right\rceil. \end{aligned}$$

Proof: Set $x = n/a$ and $m = b$ in lemma 2. ///

Exercise

Let $n \in \mathbf{Z}$. Show that (a) $\lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{2} \rfloor = n$, (b) $\lfloor \frac{n}{2} \rfloor = \lfloor \frac{n+1}{2} \rfloor$, and (c) $\lfloor \frac{n}{2} \rfloor = \lfloor \frac{n-1}{2} \rfloor$.

Logarithms

Let $x, a, b \in \mathbf{R}$ where $x > 0$, $a > 1$, and $b > 1$. Then $\log_a(x)$ denotes the exponent on a which gives x . In other words, $\log_a(x)$ is the inverse function of a^x , which means $a^{\log_a(x)} = x$ and $\log_a(a^x) = x$. Thus

$$x = a^{\log_a(x)} = \left(b^{\log_b(a)}\right)^{\log_a(x)} = b^{\log_b(a) \cdot \log_a(x)}$$

Taking \log_b of both sides of this equation yields

$$(*) \quad \log_b(x) = \log_b(a) \cdot \log_a(x),$$

which says in particular $\log_b(x) = \text{constant} \cdot \log_a(x)$, i.e. any two log functions differ by a constant multiple. It follows that $\log_b(n) = \Theta(\log_a(n))$, so speaking in terms of asymptotic growth rates, there is really only one log function. Equation (*) implies

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$$

which shows how to convert from one log function to another. In particular $\lg(x) = \frac{\ln(x)}{\ln(2)}$. Here we use the standard notation $\lg(\cdot) = \log_2(\cdot)$, and $\ln(\cdot) = \log_e(\cdot)$, where $e = 2.71828\dots$. Equation (*) also implies $a^{\log_b(x)} = a^{\log_a(x) \cdot \log_b(a)} = (a^{\log_a(x)})^{\log_b(a)} = x^{\log_b(a)}$, which gives us the useful formula

$$a^{\log_b(x)} = x^{\log_b(a)}.$$

Stirling's Formula

Let $n \in \mathbf{Z}^+$. Then $n! = \sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n \cdot \left(1 + \Theta\left(\frac{1}{n}\right)\right)$.

Stirling's formula gives a simple way to determine asymptotic (upper, lower, and tight) bounds on functions involving $n!$. An elementary proof can be found at

<http://www.sosmath.com/calculus/sequence/stirling/stirling.html>

Corollary:

- (1) $n! = o(n^n)$
- (2) $n! = \omega(b^n)$ for any $b > 0$
- (3) $\log(n!) = \Theta(n \log(n))$

Proof of (1):

$$\frac{n!}{n^n} = \frac{\sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n \cdot \left(1 + \Theta\left(\frac{1}{n}\right)\right)}{n^n} = \frac{\sqrt{2\pi n} \cdot \left(1 + \Theta\left(\frac{1}{n}\right)\right)}{e^n} \rightarrow 0 \text{ as } n \rightarrow \infty, \text{ showing that } n! = o(n^n). \quad ///$$

Proof of (3): Taking log (any base) of both sides of Stirling's formula, we get

$$\begin{aligned} \log(n!) &= \log \sqrt{2\pi n} + \log \left(\frac{n}{e}\right)^n + \log \left(1 + \Theta\left(\frac{1}{n}\right)\right) \\ &= \frac{1}{2} \log(2\pi) + \frac{1}{2} \log(n) + n \log(n) - n \log(e) + \log \left(1 + \Theta\left(\frac{1}{n}\right)\right). \end{aligned}$$

Therefore

$$\frac{\log(n!)}{n \log(n)} = 1 + (\text{stuff that } \rightarrow 0 \text{ as } n \rightarrow \infty),$$

hence $\lim_{n \rightarrow \infty} \left(\frac{\log(n!)}{n \log(n)}\right) = 1$, proving that $\log(n!) = \Theta(n \log(n))$. ///

Exercise: Prove part (2) of the corollary.

Exercise: Prove that $\binom{2n}{n} = \Theta\left(\frac{4^n}{\sqrt{n}}\right)$, where $\binom{m}{k}$ denotes the binomial coefficient $\binom{m}{k} = \frac{m!}{k!(m-k)!}$, for $0 \leq k \leq m$.

Exercise: Determine a number $a > 0$ such that $\binom{3n}{n} = \Theta\left(\frac{a^n}{\sqrt{n}}\right)$.

Exercise: Determine a number $a_k > 0$ such that $\binom{kn}{n} = \Theta\left(\frac{a_k^n}{\sqrt{n}}\right)$.