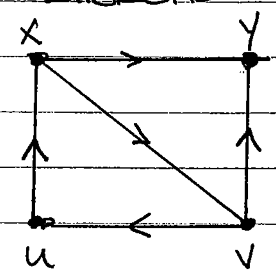
B.4 GRAPHS

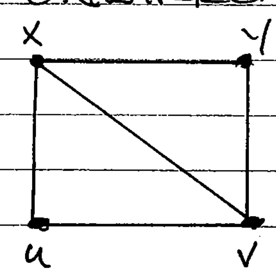
A DIRECTED GRAPH is a PAIR $G=(V,E)$ OF SETS WHERE V IS FINITE AND NON-EMPTY (CALLED VERTICES) AND $E \subset V \times V$, i.e. E CONSISTS OF ORDERED PAIRS OF VERTICES (CALLED EDGES).

IN AN UNDIRECTED GRAPH THE EDGE SET E CONSISTS OF UNORDERED PAIRS OF VERTICES.

EX DIRECTED



UNDIRECTED



$V=\{x,y,u,v\}$
$E=\{(x,y),(u,x),(v,y),(v,u),(x,v)\}$

$V=\{x,y,u,v\}$
$E=\{xy,xu,xv,uv,yv\}$

IN THE ABOVE EXAMPLES WE WOULD SAY x IS ADJACENT TO y, x IS INCIDENT WITH xy, AND xy IS ADJACENT TO xu.

THE DEGREE OF A VERTEX IS THE NUMBER OF EDGES INCIDENT WITH IT. IN A DIRECTED GRAPH WE HAVE OBVIOUS NOTIONS OF INDEGREE AND OUTDEGREE.

AN X-Y PATH IS A SEQUENCE OF VERTICES

$$P: \; x = v_0, v_1, v_2, \ldots, v_{k-1}, v_k = y$$

FOR WHICH $(v_{i-1}, v_i) \in E$ FOR $1 \leq i \leq k$. THE LENGTH OF SUCH A PATH IS THE NUMBER OF EDGES k.

WE CALL $x = v_0$ THE INITIAL VERTEX, $y = v_k$ THE TERMINAL VERTEX, AND $v_1, \ldots, v_{k-1}$ THE INTERMEDIATE OR INTERNAL VERTICES. A PATH P IS CALLED SIMPLE IF ITS INTERNAL VERTICES ARE DISTINCT.

P IS CALLED A CYCLE IF ITS INITIAL AND TERMINAL VERTICES ARE IDENTICAL (i.e. $v_0 = v_k$)

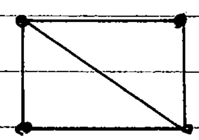
IN A DIRECTED GRAPH WE HAVE OBVIOUS NOTIONS OF DIRECTED & UNDIRECTED PATHS, CYCLES, ETC..

AN UNDIRECTED GRAPH G IS CALLED CONNECTED IF FOR ALL $x, y \in V(G)$, G CONTAINS AN X-Y PATH. A DIRECTED GRAPH IS CALLED CONNECTED IF ITS UNDERLYING UNDIRECTED GRAPH IS CONNECTED. A DIRECTED GRAPH IS CALLED STRONGLY CONNECTED IF EVERY VERTEX IS REACHABLE FROM EVERY OTHER VERTEX ALONG A DIRECTED PATH!

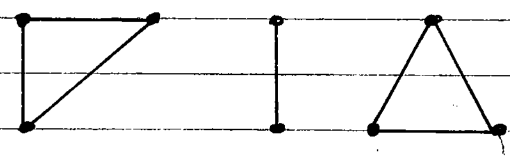
G is called <u>Acyclic</u> (also a <u>Forest</u>) if it contains no cycles.

A <u>Tree</u> is a graph which is both acyclic and connected.

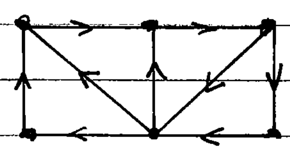
Connected



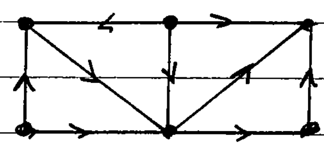
Disconnected



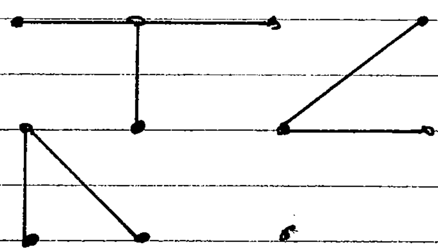
Strongly connected



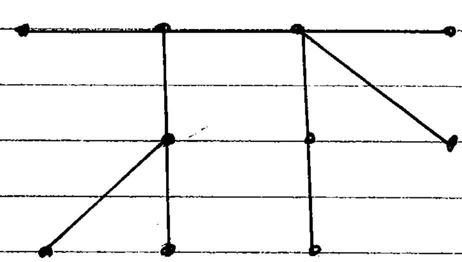
Not strongly connected.



Forest



Tree



22.1 GRAPH REPRESENTATIONS

LET G BE A GRAPH (UNDIRECTED) AND $x, y \in V(G)$. THE DISTANCE $\delta(x,y)$ FROM x TO y IS DEFINED AS

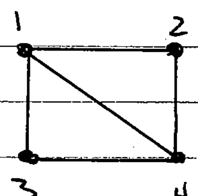
$$\delta(x,y) = \begin{cases} \text{LENGTH OF A SHORTEST } x\text{-}y \text{ PATH IF SUCH A PATH EXISTS.} \\ \infty \text{ IF NO SUCH PATH EXISTS.} \end{cases}$$

SINGLE SOURCE SHORTEST PATH PROBLEM (SSSP): GIVEN $s \in V(G)$ DETERMINE $\delta(s,y)$ FOR ALL $y \in V(G)$. AS WE'LL SEE THIS PROBLEM IS SOLVED BY THE BREADTH FIRST SEARCH (BFS) ALGORITHM.

LET $V(G) = \{v_1, v_2, \ldots, v_n\}$. THE ADJACENCY MATRIX OF G IS THE $n \times n$ MATRIX $A = A(G)$ GIVEN BY

$$A_{ij} = \begin{cases} 1 & \text{IF } (v_i, v_j) \in E \\ 0 & \text{IF } (v_i, v_j) \notin E. \end{cases}$$

EX.



$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

REMARKS

(1) $A(G)$ is NECESSARILY SYMMETRIC.

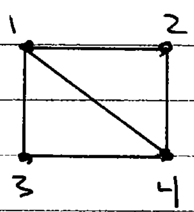
(2) THE SUM OF THE i^{TH} ROW (OR COLUMN) OF $A(G)$ is $\deg(v_i)$

(3) HANDSHAKE LEMMA

$$\sum_{i=1}^{n} \deg(v_i) = 2|E(G)|.$$

(4) THE i^{TH} ROW j^{TH} COLUMN OF A^d is THE NUMBER OF $v_i - v_j$ PATHS IN G OF LENGTH d.

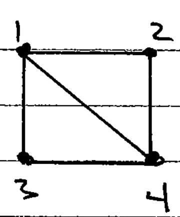
EX.



$$A^2 = \begin{pmatrix} 3 & 1 & 1 & 2 \\ 1 & 2 & 2 & 1 \\ 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 3 \end{pmatrix} \qquad A^3 = \begin{pmatrix} 4 & 5 & 5 & 5 \\ 5 & 2 & 2 & 5 \\ 5 & 2 & 2 & 5 \\ 5 & 5 & 5 & 4 \end{pmatrix}$$

THE ADJACENCY LIST REPRESENTATION OF G CONSISTS OF AN ARRAY Adj OF $n = |V(G)|$ LISTS. Adj[i] CONTAINS THE NEIGHBORS OF VERTEX v_i.

EX.



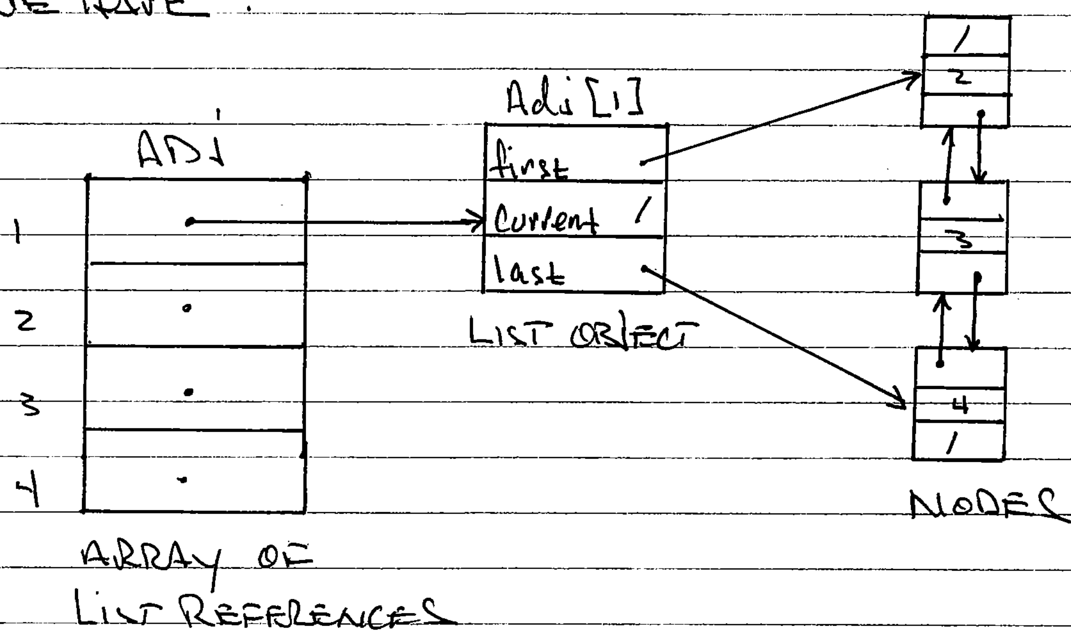
1 ¦ 2 3 4

2 ¦ 1 4

3 ¦ 1 4

4 ¦ 1 2 3

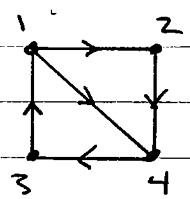
USING OUR DOUBLY LINKED LISTS OF PA1 & PA2 WE HAVE:



THE ORDER IN WHICH VERTICES ARE STORED IN Adj[i] MAY BE ARBITRARY, BUT THE PARTICULAR ORDER WILL AFFECT THE OPERATION OF BFS (AND OTHER ALGORITHMS WHICH USE ALR.)

NOTE THE ADJACENCY MATRIX AND ADJACENCY LIST REPRESENTATIONS ARE VALID ON DIRECTED GRAPHS. $A(G)$ MAY NOT BE SYMMETRIC. THE DEFINITIONS OF $\delta(x,y)$ AND SSSP GO THROUGH UNCHANGED.

Ex.



$$A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

1: 2 4
2: 4
3: 1
4: 3