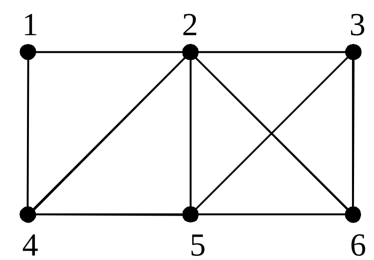
CMPS 101
Algorithms and Abstract Data Types
Graph Theory

Graphs

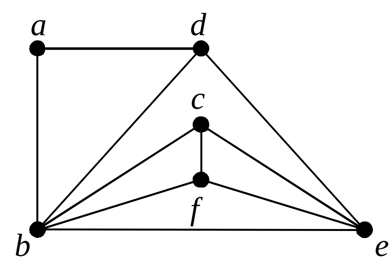
A *graph G* consists of an ordered pair of sets $G=(V,E)$ where $V\neq\varnothing$, and $E\subseteq V^{(2)}=\{2\text{-subsets of } V\}$, i.e. E consists of *unordered* pairs of elements of V. We call $V=V(G)$ the *vertex* set, and $E=E(G)$ the *edge* set of G. In this handout we consider only graphs in which both the vertex set and edge set are finite. An edge $\{x, y\}$, denoted xy or yx, is said to *join* its two *end vertices* x and y, and these ends are said to be *incident* with the edge xy. Two vertices are called *adjacent* if they are joined by an edge, and two edges are said to be *adjacent* if they have a common end vertex. A graph will usually be depicted as a collection of points in the plane (vertices), together with line segments (edges) joining the points.

Example 1 $V(G)=\{1,2,3,4,5,6\}$, $E(G)=\{12,14,23,24,25,26,35,36,45,56\}$



Two graphs G_1 and G_2 are said to me *isomorphic* if there exists a bijection $\phi: V(G_1)\to V(G_2)$ such that for any $x, y\in V(G_1)$, the pair xy is an edge of G_1 if and only if the pair $\phi(x)\phi(y)$ is an edge of G_2. In other words, ϕ must preserve all incidence relations amongst the vertices and edges in G_1. We write $G_1\cong G_2$ to mean that G_1 and G_2 are isomorphic.

Example 2 Let G_1 be the graph from the previous example, and define G_2 by $V(G_2)=\{a,b,c,d,e,f\}$, $E(G_2)=\{ab,ad,bc,bd,be,bf,ce,cf,de,ef\}$. Define a map $\phi: V(G_1)\to V(G_2)$ by $1\to a, 2\to b, 3\to c$, $4\to d$, $5\to e$, $6\to f$. Clearly ϕ is an isomorphism. G_2 can be drawn as



Isomorphic graphs are indistinguishable as far as graph theory is concerned. In fact, graph theory can be defined to be the study of those properties of graphs that are preserved by isomorphism. Thus a graph is not a picture, in spite of the way we visualize it. Instead a graph is a combinatorial object consisting of two abstract sets, together with some incidence data relating those sets.

If $x \in V(G)$ the *degree* of *x*, denoted $\deg(x)$, is the number of edges incident with vertex *x*, or equivalently, the number of vertices adjacent to *x*. Referring to Example 1 above we see that $\deg(1) = 2$, $\deg(2) = 5$, and $\deg(6) = 3$. The *degree sequence* of a graph consists of it's vertex degrees arranged in increasing order. The graph in Example 1 has degree sequence (2, 3, 3, 3, 4, 5). Observe that the graph in Example 2 has the same degree sequence. Clearly if $\phi : V(G_1) \to V(G_2)$ is an isomorphism, then $\deg(\phi(x)) = \deg(x)$ for any $x \in V(G_1)$, and hence isomorphic graphs have the same degree sequence. Observe that

$$\sum_{x \in V(G)} \deg(x) \;=\; 2\,|\,E(G)\,|$$

since each edge, having two distinct ends, contributes 2 to the sum on the left. This is sometimes known as the Handshake Lemma for it says that the number of hands shaken at a party is exactly twice the number of handshakes. It follows from this formula that the number vertices of odd degree must be even. To see this, suppose *G* contained an odd number of odd vertices. Then the left hand side of the above equation would be odd, while the right hand side is clearly even.

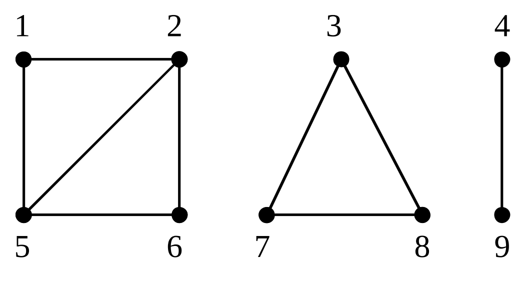
Given $x, y \in V(G)$ (not necessarily adjacent), a *walk* from *x* to y, or an *x*-*y* walk, is a sequence of vertices $x = v_0, v_1, v_2, \ldots, v_{k-1}, v_k = y$ such that $v_{i-1}v_i \in E(G)$ for $1 \le i \le k$. We call *x* the *origin* and *y* the *terminus* of the walk. These need not be distinct. If $x = y$, the walk is said to be *closed*. The *length* of the walk is *k*, the number of edge traversals performed in going from *x* to *y* along the sequence. Since the edges of a graph have no inherent direction, we do not distinguish between the above sequence and its reversal: $y = v_k, v_{k-1}, \ldots, v_2, v_1, v_0 = x$. Thus the designation as to which vertex in a walk is the origin and which is the terminus is arbitrary. A walk in which no edge is traversed more than once is called a *trail*, and a trail in which no vertex is visited more than once (except possibly when origin = terminus) is called a *path*. A closed path with at least one edge is called a *cycle*.

Example 3 Referring to the above example we have:

a cycle of length 3: 2 5 6 2
a cycle of length 6: 1 2 3 6 5 4 1
a 1-6 path of length 5: 1 4 2 5 3 6
a 1-6 path of length 2: 1 2 6
a 3-1 trail which is not a path: 3 2 5 6 2 1
a 3-1 walk which is not a trail: 3 5 2 4 5 2 1

A graph *G* is said to be *connected* if it contains an *x*-*y* path for every $x, y \in V(G)$, otherwise *G* is called *disconnected*. Examples 1 and 2 above are clearly connected, while the following is disconnected.

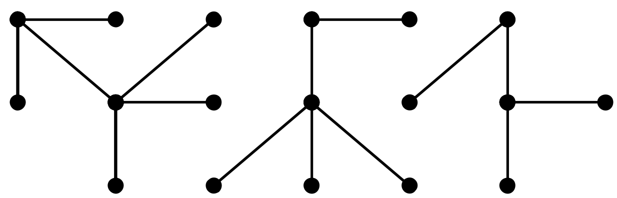
Example 4 *V* = {1, 2, 3, 4, 5, 6, 7, 8, 9} *E* = {12, 15, 25, 26, 56, 37, 38, 78, 49}



A *subgraph* of a graph G is a graph H in which $V(H) \subseteq V(G)$, and $E(H) \subseteq E(G)$. In the above example ({1, 2, 5}, {12, 15, 25}) is a connected subgraph, while ({2, 3, 6, 7}, {26, 37}) is a disconnected subgraph. A subgraph H is called a *connected component* of G if it is (i) connected, and (ii) maximal with respect to property (i), i.e. any other subgraph of G that properly contains H is disconnected. We see that Example 4 has three connected components: ({1, 2, 5, 6}, {12, 15, 25, 26, 56}), ({3, 7, 8}, {37, 38, 78}), and ({4, 9}, {49}). Obviously a graph is connected if and only if it has exactly one connected component.

Trees
A graph is called *acyclic* (also a *forest*) if it contains no cycles. A *tree* is a graph that is both connected and acyclic. The connected components of an acyclic graph are obviously trees. The following example is a forest with three connected components.

Example 5



# of vertices:	8	6	5
# of edges:	7	5	4

Observe that in each tree of this forest, the number of edges is one less that the number of vertices. This fact holds in general for all trees. The following lemmas demonstrate how the independent properties of connectedness and acyclicity are related.

Lemma 1 If T is a tree with n vertices and m edges, then $m = n-1$.
Proof: See the induction handout, example 5 page 6.

Lemma 2 If G is a connected graph with n vertices and m edges, then $m \geq n-1$.
Proof: We use induction on m, the number of edges in G, starting the induction at $m=0$. Let $G=(V,E)$ be a connected graph, and suppose $m=|E|=0$. Since G is connected we must have $n=|V|=1$, whence $m \geq n-1$, and so the base case is satisfied. Suppose $m>0$ and assume the result holds for any graph with fewer than m edges. In other words, we assume that for all graphs $G'=(V',E')$ with $|E'|<m$ that $|E'| \geq |V'|-1$. Now pick any edge e in G and remove it, and let $G-e$ denote the resulting subgraph. There are two cases to consider.

Case 1: $G-e$ is connected. In this case we apply the induction hypothesis to $G-e=(V,E-e)$ which has fewer edges than G. We conclude that $|E-e| \geq |V|-1$, so that $|E|-1 \geq |V|-1$, and therefore $m=|E| \geq |V| > |V|-1 = n-1$, as required.

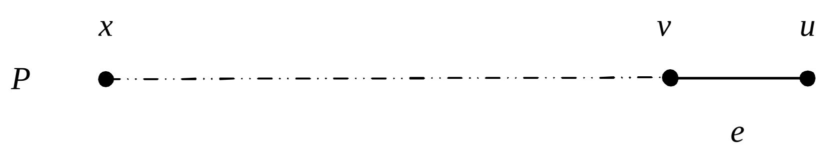
Case 2: $G-e$ is disconnected. In this case $G-e$ consists of two connected components. (**See the claim and proof below.) Call them $H_1=(V_1,E_1)$ and $H_2=(V_2,E_2)$. Note that both H_1 and H_2 have fewer edges than G, so we may apply the induction hypothesis to obtain $|E_1| \geq |V_1|-1$ and $|E_2| \geq |V_2|-1$. Therefore

$$\begin{aligned} m &= |E| = |E_1| + |E_2| + 1 \\ &\geq (|V_1| - 1) + (|V_2| - 1) + 1 \qquad \text{(by the induction hypothesis)} \\ &= |V_1| + |V_2| - 1 \\ &= |V| - 1 \qquad \text{(no vertices were removed so } |V_1| + |V_2| = |V|\text{).} \\ &= n - 1 \end{aligned}$$

The result $m \geq n-1$ now holds for all connected graphs by induction. ///

Claim**: Let G be a connected graph and $e \in E(G)$, and suppose that $G - e$ is disconnected. (Such an edge e is called a *bridge*). Then $G - e$ has exactly two connected components.

Proof: Since $G - e$ is disconnected, it has at least two components. We must show that it also has at most two components. Let e have end vertices u, and v. Let C_u and C_v be the connected components of $G - e$ that contain u and v respectively. Choose $x \in V(G)$ arbitrarily, and let P be an x-u path in G (note P exists since G is connected.) Either P includes the edge e, or it does not. If P does not contain e, then P remains intact after the removal of e, and hence P is an x-u path in $G - e$, whence $x \in C_u$. If on the other hand P does contain the edge e, then e must be the last edge along P from x to u.



In this case $P - e$ is an x-v path in $G - e$, whence $x \in C_v$. Since x was arbitrary, every vertex in $G - e$ belongs to either C_u or C_v, and therefore $G - e$ has at most two connected components. ///

Lemma 3 If G is a graph with n vertices, m edges, and k connected components, then $m \geq n - k$.
Proof: Let $H_1, H_2, \ldots, H_k$, be the connected components of G. Let n_i and m_i denote the number of vertices and edges, respectively, of H_i, for $1 \leq i \leq k$. By Lemma 2 we have $m_i \geq n_i - 1$, for $1 \leq i \leq k$, and therefore

$$m = \sum_{i=1}^{k} m_i \geq \sum_{i=1}^{k} (n_i - 1) = \sum_{i=1}^{k} n_i - \sum_{i=1}^{k} 1 = n - k. \qquad ///$$

Lemma 4 If G is a forest with n vertices, m edges, and k connected components, then $m = n - k$.
Proof: Exercise. (Hint: Emulate the proof of Lemma 3 and use Lemma 1.)

Lemma 5 Let G be a connected graph with n vertices and m edges. Suppose also that $m = n - 1$. Then G is acyclic, and hence a tree.
Proof: Suppose G is connected and $m = n - 1$. Assume, to get a contradiction, that G is not acyclic. Let e be any edge belonging to any cycle in G. Remove e from G, and denote the resultant graph by $G - e$. Then $|E(G - e)| = m - 1 < m = n - 1$. However, since e is a cycle edge, its removal does not disconnect G, so $G - e$ is also connected. Lemma 2 above, applied to $G - e$, gives

$|E(G-e)| \geq |V(G-e)|-1 = n-1$, and hence $|E(G-e)| < n-1 \leq |E(G-e)|$, which is absurd. This contradiction shows that our assumption was false, and therefore G is acyclic. ///

Lemma 6 Let G be an acyclic graph with n vertices and m edges. Suppose also that $m = n-1$. Then G is connected, and hence a tree.
Proof: Suppose G is acyclic and $m = n-1$. Let k be the number of connected components belonging to G. By Lemma 4 we have $m = n-k$, whence $n-1 = n-k$, and therefore $k = 1$, showing that G is connected. ///

Consider the following three properties of a graph $G = (V, E)$ in light of Lemmas 1, 5, and 6:

(i) G is connected,
(ii) G is acyclic
(iii) $|E| = |V|-1$.

We see that these properties are logically dependent in the sense that if any two hold, then the third must also hold. Lemma 1 states that (i) and (ii) together imply (iii), Lemma 5 says that (i) and (iii) imply (ii), and Lemma 6 says (ii) and (iii) imply (i). The following theorem summarizes these and other facts about trees.

Theorem 1 (The Treeness Theorem) Let $G = (V, E)$ be a graph. Then the following statements are logically equivalent.

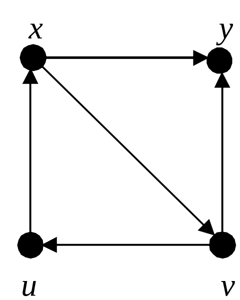
a) G is a tree (i.e. G is connected and acyclic).
b) G contains a *unique* x-y path for any $x, y \in V$.
c) G is connected, but if any edge is removed from E, the resulting graph is disconnected.
d) G is connected, and $|E| = |V|-1$.
e) G is acyclic, and $|E| = |V|-1$.
f) G is acyclic, but if any edge is added to E (joining two non-adjacent vertices), then the resulting graph contains a *unique* cycle.

Proof: As mentioned in the preceding paragraph, Lemmas 1, 5, and 6 have already established the equivalences $(a) \Leftrightarrow (d) \Leftrightarrow (e)$.

Directed Graphs

A *Directed Graph* (or *Digraph*) $G = (V, E)$ is a pair of sets, where the vertex set $V = V(G)$ is, as before, finite and non-empty, and the edge set $E = E(G) \subseteq V \times V$, i.e. E consists of *ordered* pairs of vertices.

Example 6 $V = \{x, y, u, v\}$ and $E = \{(x, y), (u, x), (v, y), (v, u), (x, v)\}$



The directed edge (x, y) in the above example is said to have *origin* x and *terminus* y, and we say that x is *adjacent* to y. The origin and terminus of a directed edge are said to be *incident* with that edge. Two edges are called *adjacent* if they have a common end vertex, so for instance (x, y) in the above example is adjacent to (u, x). The *in degree* of a vertex is the number of edges having that vertex as terminus, and it's *out degree* is the number of edges having that vertex as origin. The *degree* of a vertex is the sum if it's in degree and out degree. Thus in the above example $\mathrm{id}(x)=1$, $\mathrm{od}(x)=2$, and $\deg(x)=3$. The analog of the handshake lemma for directed graphs is

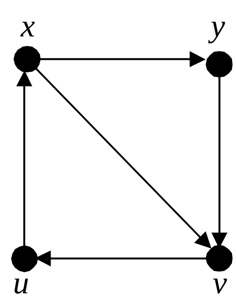
$$\sum_{x \in V(G)} \mathrm{id}(x) = \sum_{x \in V(G)} \mathrm{od}(x) = |E(G)|$$

As in the undirected case, there is a simple notion of isomorphism for directed graphs. Two digraphs G_1 and G_2 are said to me *isomorphic* if there exists a bijection $\phi: V(G_1) \to V(G_2)$ such that for any $x, y \in V(G_1)$, the ordered pair (x, y) is a directed edge of G_1 if and only if the ordered pair $(\phi(x), \phi(y))$ is a directed edge of G_2. Thus ϕ preserves incidence relations and directionality amongst the vertices and edges of G_1. We write $G_1 \cong G_2$ to mean that G_1 and G_2 are isomorphic.

A *directed path* P in a digraph is a finite sequence of vertices P: $v_0, v_1, v_2, \ldots, v_{k-1}, v_k$ such that $(v_{i-1}, v_i) \in E$ for all $1 \le i \le k$. As in the undirected case, we require that all vertices be distinct (except possibly v_0 and v_k), and that no edge be traversed more than once. If it so happens that the initial and terminal vertices are the same, $v_0 = v_1$, the path is called a *directed cycle*. The length of such a path is k, the number of edges traversed. If $x = v_0 \ne v_k = y$, we call P a directed x-y path. Notice that, unlike the undirected case, a directed x-y path and a directed y-x path are not the same thing. We say that $y \in V(G)$ is *reachable* from $x \in V(G)$ if G contains a directed x-y path. Observe that the reachability relation is reflexive (x is reachable from x via the trivial path with no edges), transitive (if y is reachable from x, and z is reachable from y, then z is reachable from x), but not symmetric (y may be reachable from x without x being reachable from y).

A digraph G is said to be *strongly connected* if for all $x, y \in V(G)$, both x is reachable from y, and y is reachable from x. Notice that the digraph in Example 6 above is not strongly connected, since for instance, u is not reachable from y. The following example *is* strongly connected.

Example 7 $V = \{x, y, u, v\}$ and $E = \{(x, y), (u, x), (y, v), (v, u), (x, v)\}$



More generally, a subset $S \subseteq V(G)$ is said to be *strongly connected* if for all $x, y \in S$, both x is reachable from y, and y is reachable from x. Furthermore, a subset $S \subseteq V(G)$ is said to be a *strongly connected* *component* of the digraph G if it is (i) strongly connected, and (ii) maximal with respect to property (i), i.e. any other subset of $V(G)$ that properly contains S is not strongly connected. Obviously G is strongly connected iff it has just one strongly connected component, namely $V(G)$ itself. Going back to the digraph in Example 6, we see that it has 2 strongly connected components: $\{x, u, v\}$ and $\{y\}$.

Representations of Graphs

We discuss three methods for representing graphs and digraphs in terms of standard data structures available in most computer languages. They are called the *Incidence Matrix*, the *Adjacency Matrix*, and the *Adjacency List* representations respectively.

The *Incidence Matrix* $I(G)$ requires that both the vertex set $V(G)$ and the edge set $E(G)$ be ordered. For this purpose we suppose that $V = \{x_1, x_2, \ldots, x_n\}$ and $E = \{e_1, e_2, e_3, \ldots, e_m\}$. Then $I(G)$ is an $n \times m$ rectangular matrix. Row i corresponds to vertex x_i, for $1 \le i \le n$. Column j corresponds to edge e_j ($1 \le j \le m$), and contains zeros everywhere except for the two rows corresponding to the ends of e_j. If G is an undirected graph, these two rows contain 1s. If G is a directed graph, the row corresponding to the origin of e_j contains -1, while the row corresponding to the terminus of e_j contains $+1$. Thus $I(G) = (I_{ij})$ where in the undirected case:

$$I_{ij} = \begin{cases} 1 & \text{if } x_i \text{ is incident with } e_j \\ 0 & \text{otherwise} \end{cases}$$

and in the directed case:

$$I_{ij} = \begin{cases} 1 & \text{if } x_i \text{ is the terminus of } e_j \\ -1 & \text{if } x_i \text{ is the origin of } e_j \\ 0 & \text{otherwise} \end{cases}$$

The *Adjacency Matrix* $A(G)$ requires that only the vertex set come equipped with an order. It is a square matrix of size $n \times n$, where $n = |V|$.