## 3.2 SOME COMMON FUNCTIONS

### FLOORS & CEILINGS

GIVEN $x \in \mathbb{R}$, $\lfloor x \rfloor$ AND $\lceil x \rceil$ DENOTE THE UNIQUE INTEGERS SATISFYING

$$x-1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x+1$$

EQUIVALENTLY IF $x \in \mathbb{R}$ AND $N \in \mathbb{Z}$:

(1) $N = \lfloor x \rfloor$ IFF $N \leq x < N+1$

(2) $N = \lceil x \rceil$ IFF $N-1 < x \leq N$

i.e. $\lfloor x \rfloor$ IS THE GREATEST INTEGER LESS THAN OR EQUAL TO $x$, AND $\lceil x \rceil$ IS THE LEAST INTEGER GREATER THAN OR EQUAL TO $x$.

### LEMMA 1

LET $x \in \mathbb{R}$, $a, b \in \mathbb{Z}$. THEN

(1) $a \leq x < b$ IFF $a \leq \lfloor x \rfloor < b$

(2) $a < x \leq b$ IFF $a < \lceil x \rceil \leq b$.

PROOF OF (1)

(i) $a \leq x \Rightarrow a \leq \lfloor x \rfloor$ SINCE AMONGST ALL INTEGERS WHICH ARE LESS THAN OR EQUAL TO $x$, $\lfloor x \rfloor$ IS THE GREATEST.

(ii) $x < b \Rightarrow \lfloor x \rfloor < b$ SINCE $\lfloor x \rfloor \leq x$.

(iii) $a \leq \lfloor x \rfloor \Rightarrow a \leq x$ SINCE $\lfloor x \rfloor \leq x$.

(iv) $\lfloor x \rfloor < b \Rightarrow x < b$ SINCE $b \leq x \Rightarrow b \leq \lfloor x \rfloor$ BY (i). ///

LEMMA 2
LET $x \in \mathbb{R}$, $m \in \mathbb{Z}^+$. THEN

(1) $\lfloor \lfloor x \rfloor / m \rfloor = \lfloor x/m \rfloor$

(2) $\lceil \lceil x \rceil / m \rceil = \lceil x/m \rceil$

PROOF OF (1)
LET $N = \lfloor \lfloor x \rfloor / m \rfloor$. THEN

$$N \le \frac{\lfloor x \rfloor}{m} < N+1$$

$\therefore\ mN \le \lfloor x \rfloor < m(N+1)$

$\therefore\ mN \le x < m(N+1)$ (BY LEMMA 1.)

$\therefore\ N \le x/m < N+1$

$\therefore\ N = \lfloor x/m \rfloor$. ///.

LEMMA 3
LET $a, b, n \in \mathbb{Z}^+$. THEN

(1) $\lfloor \lfloor n/a \rfloor / b \rfloor = \lfloor n/ab \rfloor$

(2) $\lceil \lceil n/a \rceil / b \rceil = \lceil n/ab \rceil$

PROOF OF (1).
SET $x = n/a$ AND $m = b$ IN LEMMA 2. ///.

EXERCISE

PROVE PARTS (2) OF LEMMAS 1, 2, & 3.

EXERCISE

PROVE IF $n \in \mathbb{Z}$, THEN $\lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil = n$.
ALSO SHOW $\lceil n/2 \rceil = \lfloor \frac{n+1}{2} \rfloor$ AND $\lfloor \frac{n}{2} \rfloor = \lceil \frac{n-1}{2} \rceil$.

LOGARITHMS

LET $x, a, b \in \mathbb{R}$, $x > 0$, $a > 1$, $b > 1$. THEN $\log_a(x)$ DENOTES THE EXPONENT ON $a$ WHICH GIVES $x$. THUS

$$x = a^{\log_a(x)} = \left(b^{\log_b(a)}\right)^{\log_a(x)} = b^{\log_b(a) \cdot \log_a(x)}$$

(*) $\therefore \log_b(x) = \log_b(a) \cdot \log_a(x)$

$\therefore \log_b(n) = \text{CONST} \cdot \log_a(n)$

i.e. ANY TWO log FUNCTIONS DIFFER BY A CONSTANT MULTIPLE, WHENCE

$$\boxed{\log_b(n) = \Theta(\log_a(n))}$$

EQUATION (*) GIVES

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)} \qquad \therefore \lg(x) = \frac{\ln(x)}{\ln(2)}$$

WHICH SHOWS HOW TO CONVERT FROM ONE log TO ANOTHER.

EQUATION (*) ALSO IMPLIES

$$a^{\log_b(x)} = a^{\log_a(x) \cdot \log_b(a)} = \left(a^{\log_a(x)}\right)^{\log_b(a)} = x^{\log_b(a)}$$

$$\therefore \quad \boxed{a^{\log_b(x)} = x^{\log_b(a)}}$$

STIRLING'S FORMULA

LET $n \in \mathbb{Z}^+$. THEN

$$\boxed{n! = \sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n \cdot \left(1 + \Theta\left(\frac{1}{n}\right)\right)}$$

COROLLARY

(1) $n! = o(n^n)$

(2) $n! = \omega(2^n)$

(3) $\log(n!) = \Theta(n \log n)$

PROOF OF (1)

$$\frac{n!}{n^n} = \frac{\sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n \cdot \left(1 + \Theta\left(\frac{1}{n}\right)\right)}{n^n}$$

$$= \frac{\sqrt{2\pi n}\cdot\left(1+\Theta\left(\frac{1}{n}\right)\right)}{e^n} \to 0 \text{ AS } n \to \infty.$$

$\therefore\ n! = o(n^n)$, AS CLAIMED.

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PROOF OF (3)

$$\log(n!) = \log\sqrt{2\pi n} + \log\left(\frac{n}{e}\right)^n + \log\left(1+\Theta\left(\frac{1}{n}\right)\right)$$

$$= \frac{1}{2}\log 2\pi + \frac{1}{2}\log n + n\log n - n\log e + \log\left(1+\Theta\left(\frac{1}{n}\right)\right)$$

$$\therefore \frac{\log(n!)}{n\log n} = 1 + \left(\begin{array}{c}\text{STUFF WHICH} \to 0 \\ \text{AS } n \to \infty\end{array}\right)$$

$\therefore\ \log(n!) = \Theta(n\log n)$.

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EXERCISE: PROVE (2)

EXERCISE: PROVE THAT

$$\binom{2n}{n} = \Theta\left(\frac{4^n}{\sqrt{n}}\right)$$

READ INDUCTION HANDOUT.