EX. LET $A_1, A_2, A_3 \ldots$ BE SETS. THEN

$$\forall n \geq 1 : \quad \overline{\bigcup_{k=1}^{n} A_k} = \bigcap_{k=1}^{n} \overline{A_k} .$$

PROOF:
LET $P(n)$ DENOTE THE ABOVE FORMULA.

I. $P(1)$ IS $\overline{A_1} = \overline{A_1}$, WHICH IS OBVIOUSLY TRUE ✓

II. LET $n \geq 1$ AND ASSUME $P(n)$, i.e. FOR THIS n ASSUME

$$\overline{\bigcup_{k=1}^{n} A_k} = \bigcap_{k=1}^{n} \overline{A_k}$$

WE MUST SHOW $P(n+1)$ IS TRUE, i.e.

$$\overline{\bigcup_{k=1}^{n+1} A_k} = \bigcap_{k=1}^{n+1} \overline{A_k} .$$

OBSERVE

$$\begin{aligned}
\overline{\bigcup_{k=1}^{n+1} A_k} &= \overline{\left(\bigcup_{k=1}^{n} A_k\right) \cup A_{n+1}} \\
&= \overline{\bigcup_{k=1}^{n} A_k} \cap \overline{A_{n+1}} \quad \text{(BY DEMOR LAW)} \\
&= \left(\bigcap_{k=1}^{n} \overline{A_k}\right) \cap \overline{A_{n+1}} \quad \text{(BY IND. HYP.)} \\
&= \bigcap_{k=1}^{n+1} \overline{A_k} .
\end{aligned}$$

$\therefore P(n+1)$ is TRUE

$\therefore \forall n \geq 1: \overline{\bigcup_{k=1}^{n} A_k} = \bigcap_{k=1}^{n} \overline{A_k}$. ///

THE PROOF OF PMI RELIES ON THE FOLLOWING FACT, KNOWN AS THE <u>WELL ORDERING PROPERTY</u> OF $\mathbb{Z}^+$:

- ANY NON-EMPTY SUBSET OF $\mathbb{Z}^+$ CONTAINS A LEAST ELEMENT.

<u>THEOREM</u> (PMI)
FOR ANY PROPOSITIONAL FUNCTION $P: \mathbb{Z}^+ \to \{T, F\}$, THE FOLLOWING IS A TAUTOLOGY

$$\left[P(1) \wedge \forall n\, (P(n) \to P(n+1))\right] \to \forall n\, P(n).$$

<u>PROOF</u>:
LET $P: \mathbb{Z}^+ \to \{T, F\}$ BE A PROPOSITIONAL FUNCTION, AND ASSUME $P(1)$ AND $\forall n\, (P(n) \to P(n+1))$ ARE TRUE. WE MUST SHOW $\forall n\, P(n)$ IS TRUE.

LET $S = \{n \in \mathbb{Z}^+ \mid \neg P(n)\}$, i.e. S is THE SET OF POSITIVE INTEGERS FOR WHICH

$P(n)$ is false. It will be sufficient to show $S = \emptyset$, for then $P(n)$ must be true for all $n \in \mathbb{Z}^+$.

Assume, to get a contradiction, that $S \neq \emptyset$. By the well ordering property S contains a smallest element, call it m. Thus $m \in S$ but $m-1 \notin S$.

Since $P(1)$ is true we have $1 \notin S$, so that $m \neq 1$, and hence $m-1 \in \mathbb{Z}^+$. Since $\forall n\,(P(n) \to P(n+1))$ is true we have $P(m-1) \to P(m)$, by universal instantiation. But recall $m-1 \notin S$ so that $P(m-1)$ is true. It follows that $P(m)$ is also true by modus ponens, and therefore $m \notin S$.

We now have the contradiction that $m \in S$ and $m \notin S$, showing that $S = \emptyset$ as required.

///.

There are many variations on the induction technique.

THE INDUCTION STEP CAN BE RE-PARAMETRIZED AS FOLLOWS:

I. SHOW $P(n_0)$

IIa. SHOW $\forall n > n_0 : P(n-1) \Rightarrow P(n)$

EX $\forall n \geq 1 : \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$. (AGAIN.)

PROOF: LET $P(n)$ BE THE ABOVE FORMULA.

I. $P(1)$ REDUCES TO $1=1$.

II. LET $n > 1$ AND ASSUME $P(n-1)$ IS TRUE. i.e. ASSUME

$$\sum_{k=1}^{n-1} k = \frac{(n-1)\cdot n}{2}.$$

THEN

$$\sum_{k=1}^{n} k = \left(\sum_{k=1}^{n-1} k\right) + n$$

$$= \frac{n(n-1)}{2} + n \quad \text{(BY IND. HYP.)}$$

$$= \frac{n(n-1)+2n}{2}$$

$$= \frac{n(n+1)}{2}$$

$\therefore P(n)$ IS TRUE

$\therefore$ RESULT FOLLOWS BY PMI. ///

ANOTHER VARIATION OF THE INDUCTION TECHNIQUE IS CALLED STRONG INDUCTION. (ALSO THE 2^{ND} PRINCIPLE OF MATHEMATICAL INDUCTION.) SEE P. 249 IN TEXT.

I. PROVE $P(1)$

II. PROVE $\forall n > 1 : (P(1) \wedge P(2) \wedge \cdots \wedge P(n-1)) \rightarrow P(n)$

IN TERMS OF THE DOMINO ANALOGY, WE ASSUME IN THE INDUCTION STEP THAT ALL DOMINOS UP TO, BUT NOT INCLUDING THE n^{TH} DOMINO FALL, THEN SHOW AS A CONSEQUENCE THAT THE n^{TH} DOMINO FALLS.

IN THIS CASE THE PHRASE 'INDUCTION HYPOTHESIS' REFERS TO THE STRONGER ASSUMPTION $P(1) \wedge \cdots \wedge P(n)$. WE CAN ALSO WRITE THIS AS

II. $\forall n > 1 : (\forall k < n : P(k)) \rightarrow P(n)$.

THEOREM

EVERY POSITIVE INTEGER CAN BE WRITTEN AS THE PRODUCT OF ZERO OR MORE PRIMES.

NOTE: THIS IS HALF OF THE FTA. THE OTHER HALF IS UNIQUENESS.

PROOF:

LET $P(n)$ = 'n IS THE PRODUCT OF PRIMES'

I. $P(1)$ IS TRUE SINCE 1 IS THE EMPTY PRODUCT. (IF THIS IS UNSATISFYING START INDUCTION AT $n_0 = 2$. $P(2)$ IS CERTAINLY TRUE.)

II. LET $n > 1$ AND ASSUME $P(1), P(2), \ldots, P(n-1)$ ARE ALL TRUE. i.e. $P(k)$ IS TRUE FOR $1 \leq k < n$. i.e. ANY k IN THE RANGE $1 \leq k < n$ IS THE PRODUCT OF PRIMES.

IF n IS PRIME, THEN $P(n)$ IS TRUE AND WE'RE DONE. OTHERWISE n IS COMPOSITE, SO THAT $n = a \cdot b$ WHERE $1 < a < n$ AND $1 < b < n$. BY THE INDUCTION HYPOTHESIS BOTH $P(a)$ AND $P(a)$ ARE TRUE, i.e. BOTH a AND b ARE THE PRODUCT OF PRIMES. THEREFORE $n = a \cdot b$ CAN BE WRITTEN AS A PRODUCT OF PRIMES. $\therefore P(n)$ IS TRUE.

$\therefore \forall n \geq 1$: n IS A PRODUCT OF PRIMES.

///