(3.2) SEQUENCES

DEFN

A SEQUENCE IS A FUNCTION WHOSE DOMAIN IS A SUBSET OF $\mathbb{Z}$.

OFTEN THE DOMAIN IS $\mathbb{Z}^+$ OR $\mathbb{N}$. IF THE DOMAIN IS FINITE WE SPEAK OF A FINITE SEQUENCE.

WE WRITE a_n (OR b_n, OR c_n, ETC.) FOR THE IMAGE OF n UNDER SUCH A FUNCTION. a_n IS CALLED THE n^{TH} TERM OF THE SEQUENCE.

NOTATION: $(a_n)_{n=1}^{\infty}$ OR $(a_n)_{n=0}^{\infty}$

EX. $\left(\frac{1}{n}\right)_{n=1}^{\infty} = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right)$

EX. $(n^2)_{n=0}^{\infty} = (0, 1, 4, 9, 16, \ldots)$

EX $\left(\frac{n}{n+1}\right)_{n=0}^{\infty} = \left(0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\right)$

EX. $(2^n)_{n=0}^{\infty} = (1, 2, 4, 8, \ldots)$

A SEQUENCE IS OFTEN THOUGHT OF AS SIMPLY A LIST (USUALLY INFINITE)

A GEOMETRIC SEQUENCE (OR PROGRESSION) IS A SEQUENCE OF THE FORM

$$(ar^n)_{n=0}^{\infty} = (a, ar, ar^2, ar^3, \dots)$$

a IS CALLED THE INITIAL TERM, r IS CALLED THE COMMON RATIO.

AN ARITHMETIC SEQUENCE (OR PROGRESSION) IS A SEQUENCE OF THE FORM

$$(a+nd)_{n=0}^{\infty} = (a, a+d, a+2d, a+3d, \dots)$$

a IS THE INITIAL TERM, d THE COMMON DIFFERENCE.

A SUMMATION (OR SERIES) IS THE SUM OF A SEQUENCE, WHICH MAY BE FINITE OR INFINITE.

$$\sum_{j=1}^{n} a_j = a_1 + a_2 + \cdots + a_n$$

OR

$$\sum_{j=m}^{n} a_j = a_m + a_{m+1} + \cdots + a_n \qquad (m \le n)$$

j IS THE INDEX OF SUMMATION. m AND n ARE THE LOWER AND UPPER LIMITS OF SUMMATION RESPECTIVELY.

Ex. $$\sum_{k=1}^{5} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$$

$$\sum_{j=0}^{6} 2^j = 1 + 2 + 4 + 8 + 16 + 32 + 64$$

$$\sum_{n=1}^{100} \frac{n}{n+1} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{100}{101}$$

OTHER NOTATION: IF $S \subseteq \mathbb{Z}$ WE WRITE

$$\sum_{k \in S} a_k$$

FOR THE SUM OF ALL TERMS a_k WITH $k \in S$.

Ex. $$\sum_{k \in \{2,4,6\}} k^2 = 2^2 + 4^2 + 6^2$$

SOME SPECIAL SUMMATION FORMULAS

(1) $$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

(2) $$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

(3) $$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

(4) $$\sum_{k=0}^{n} a x^k = \begin{cases} a\left(\dfrac{x^{n+1}-1}{x-1}\right) & \text{IF } x \neq 1 \\ a \cdot (n+1) & \text{IF } x = 1 \end{cases}$$

PROOF OF (4)

THE CASE $x=1$ IS OBVIOUS, SO SUPPOSE $x \neq 1$. LET

$$S = a + ax + ax^2 + \cdots + ax^n$$
$$\therefore \; xS = \quad ax + ax^2 + \cdots + ax^n + ax^{n+1}$$
$$\therefore \; xS - S = ax^{n+1} - a$$
$$\therefore \; (x-1)S = a(x^{n+1}-1)$$
$$\therefore \; S = a\left(\frac{x^{n+1}-1}{x-1}\right). \qquad ///$$

EX.
$$\sum_{k=0}^{20} 3\cdot 2^k = 3\left(\frac{2^{21}-1}{2-1}\right) = 3\cdot(2^{21}-1) = 6291453$$
$$\sum_{k=1}^{1000} k = \frac{1000\cdot 1001}{2} = 500\cdot 1001 = 500500$$
$$\sum_{k=1}^{10} k^2 = \frac{10\cdot 11\cdot 21}{6} = 385$$

CARDINALITY OF SETS:

TWO SETS A, B (NOT NECESSARILY FINITE) ARE SAID TO HAVE THE SAME CARDINALITY IF THERE EXISTS A BIJECTION FROM ONE TO THE OTHER:

$$f : A \to B$$

NOTATION: WE WRITE $|A|=|B|$ TO MEAN A AND B HAVE THE SAME CARDINALITY

WE HAVE EXTENDED THE NOTION OF TWO SETS HAVING THE 'SAME NUMBER OF ELEMENTS' TO INFINITE SETS.

EX. LET $E=\{n\in\mathbb{N} \mid n \text{ is even}\}$ AND DEFINE $f:\mathbb{N}\to E$ BY

$$f(n) = 2n$$

ONE CHECKS EASILY THAT f IS A BIJECTION, SO $|E|=|\mathbb{N}|$! THIS IN SPITE OF THE FACT THAT E IS A PROPER SUBSET OF $\mathbb{N}$.

EX DEFINE $f:\mathbb{N}\to\mathbb{Z}$ BY THE RULE:

$$\begin{array}{ccc} \mathbb{N} & & \mathbb{Z} \\ 0 & \to & 0 \\ 1 & \to & 1 \\ 2 & \to & -1 \\ 3 & \to & 2 \\ 4 & \to & -2 \\ \vdots & & \vdots \end{array} \qquad \text{i.e. } f(n) = \begin{cases} -\frac{n}{2} & \text{if } n \text{ even} \\ \frac{n+1}{2} & \text{if } n \text{ odd} \end{cases}$$

OBVIOUSLY f IS A BIJECTION, SO $|\mathbb{N}|=|\mathbb{Z}|$.

DEFN

A SET IS CALLED <u>COUNTABLE</u> IF IT IS EITHER FINITE, OR HAS THE SAME CARDINALITY AS $\mathbb{N}$ (OR $\mathbb{Z}^+$). A SET WHICH IS NOT COUNTABLE IS CALLED <u>UNCOUNTABLE</u>.

WE'VE SEEN $|\mathbb{N}| = |E| = |\mathbb{Z}|$, SO ALL THESE SETS ARE COUNTABLE. MORE PRECISELY THESE SETS ARE SAID TO BE <u>COUNTABLY INFINITE</u>. NOTE OBVIOUSLY $|\mathbb{Z}^+| = |\mathbb{N}|$.

INFORMALLY A SET IS <u>COUNTABLE</u> IF ITS ELEMENTS CAN BE PLACED IN AN ORDERED LIST, i.e. A SEQUENCE (FINITE OR INFINITE.)

<u>EX.</u> $\mathbb{Z}^+ \times \mathbb{Z}^+$ IS COUNTABLE

$$\begin{array}{ccccc}
11 & 12 & 13 & 14 & \cdots \\
21 & 22 & 23 & 24 & \cdots \\
31 & 32 & 33 & 34 & \cdots \\
41 & 42 & 43 & 44 & \cdots \\
\vdots & \vdots & \vdots & \vdots &
\end{array}$$

<u>EXERCISE</u>: FIND AN EXPLICIT FORMULA FOR THE BIJECTION $f: \mathbb{Z}^+ \to \mathbb{Z}^+ \times \mathbb{Z}^+$ ABOVE. (i.e. $1 \to 11,\ 2 \to 21,\ 3 \to 12,\ 4 \to 31, \ldots$)

EXERCISE

SHOW THAT ANY SUBSET OF A COUNTABLE SET IS COUNTABLE. HINT: IT'S SUFFICIENT TO SHOW ANY SUBSET OF $\mathbb{Z}^+$ IS COUNTABLE

EXERCISE

SHOW THAT $\mathbb{Q}$ IS COUNTABLE. HINT: SHOW THAT THERE IS A BIJECTION FROM $\mathbb{Q}^+$ (THE POSITIVE RATIONALS) TO A CERTAIN SUBSET OF $\mathbb{Z}^+ \times \mathbb{Z}^+$.

THEOREM

$\mathbb{R}$ IS NOT COUNTABLE.

PROOF:

WE ASSUME $\mathbb{R}$ IS COUNTABLE AND DERIVE A CONTRADICTION. IF $\mathbb{R}$ IS COUNTABLE THEN SO IS THE SUBSET

$$S = \{ x \in \mathbb{R} \mid 0 < x \leq 1 \}$$

SINCE S IS (SUPPOSEDLY) COUNTABLE, ITS MEMBERS CAN BE MADE TO FORM AN INFINITE LIST OR SEQUENCE

$$S = \{ r_1, r_2, r_3, \ldots\ldots \}$$

EACH ELEMENT OF S HAS A DECIMAL EXPANSION

$$\begin{aligned} r_1 &= .d_{11} d_{12} d_{13} \ldots \\ r_2 &= .d_{21} d_{22} d_{23} \ldots \\ r_3 &= .d_{31} d_{32} d_{33} \ldots \\ &\vdots \end{aligned}$$

WHERE EACH DIGIT d_{ij} IS IN THE SET $\{0, 1, 2, \ldots, 9\}$.

(IF THERE ARE TWO DECIMAL EXPANSIONS, CHOOSE THE ONE IN WHICH THERE IS NO INFINITE SEQUENCE OF 0'S. i.e. REPRESENT .5 AS .499... RATHER THAN .500...)

NOW THE ABOVE LIST SUPPOSEDLY CONTAINS <u>ALL</u> ELEMENTS OF S. DEFINE $x \in \mathbb{R}$ BY

$$x = .c_1 c_2 c_3 \ldots$$

WHERE WE PICK $c_i \in \{1, 2, \ldots, 8\}$ SUCH THAT $c_i \neq d_{ii}$ FOR $i = 1, 2, 3, \ldots$

OBSERVE THAT $x \in S$ BY IT'S VERY CONSTRUCTION.

BUT $x \neq r_1$ SINCE $c_1 \neq d_{11}$. LIKEWISE $x \neq r_2$ SINCE $c_2 \neq d_{22}$. SIMILARLY $x \neq r_3$, $x \neq r_4, \ldots$. IN FACT x DOES NOT EQUAL ANY r_i IN THE LIST. $\therefore x \notin S$.

THIS CONTRADICTION SHOWS THAT OUR ASSUMPTION WAS FALSE. $\therefore \mathbb{R}$ IS NOT COUNTABLE.

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REMARKS

- THIS FAMOUS PROOF IS CALLED THE CANTOR DIAGONAL ARGUMENT AFTER GEORG CANTOR WHO FIRST DISCOVERED IT.

- THIS RESULT TELLS US THAT THERE IS MORE THAN ONE KIND OF INFINITY!

$$|\mathbb{R}| \neq |\mathbb{Z}^+|$$