

# 4.2 THE PIGEONHOLE PRINCIPLE

## EX.

A DRAWER CONTAINS A DOZEN BROWN SOCKS AND A DOZEN BLACK SOCKS (UNMATCHED.)  
 A MAN TAKES OUT SOCKS AT RANDOM IN THE DARK. HOW MANY SOCKS MUST BE REMOVED TO BE SURE HE HAS (AT LEAST) 2 OF THE SAME COLOR?  
 ANSWER: 3

WHAT IS THE SMALLEST NUMBER OF SOCKS WHOSE REMOVAL WOULD GUARANTEE THAT THE MAN HAS 6 OF THE SAME COLOR?  
 ANSWER: 11

THEOREM : PIGEONHOLE PRINCIPLE (PP)  
 IF  $k+1$  (OR MORE) OBJECTS ARE PLACED IN  $k$  BOXES, THEN (AT LEAST) ONE BOX CONTAINS (AT LEAST) 2 OBJECTS.

## PROOF

ASSUME, TO GET A CONTRADICTION, THAT ALL  $k$  BOXES CONTAIN FEWER THAN 2 OBJECTS.  
 ∴ ALL BOXES CONTAIN AT MOST 1 OBJECT  
 ∴ THE NUMBER OF OBJECTS IS AT MOST THE NUMBER OF BOXES  $k$ . THIS CONTRADICTS THAT THERE ARE  $k+1$  OBJECTS.    ///

Ex.

IN ANY GROUP OF 13 PEOPLE, AT LEAST 2 WERE BORN IN THE SAME MONTH.

Ex.

HOW MANY STUDENTS MUST BE ENROLLED IN A UNIVERSITY TO GUARANTEE THAT AT LEAST 100 WERE BORN IN THE SAME MONTH?

ANSWER: 1189

THEOREM: GENERALIZED PIGEONHOLE PRINCIPLE (GPP)

IF  $n$  (OR MORE) OBJECTS ARE PLACED IN  $k$  BOXES, THEN (AT LEAST) ONE BOX CONTAINS (AT LEAST)  $\lceil \frac{n}{k} \rceil$  OBJECTS.

IN THE PREVIOUS EXAMPLE, WE SEEK THE SMALLEST  $n$  FOR WHICH

$$\lceil \frac{n}{12} \rceil = 100$$

i.e.  $99 < \frac{n}{12} \leq 100$

i.e.  $12 \cdot 99 < n \leq 12 \cdot 100$ .

THE SMALLEST SUCH  $n$  IS OBVIOUSLY  $n = 12 \cdot 99 + 1 = 1189$ .

PROOF: (OF GPP)

ASSUME, TO GET A CONTRADICTION, THAT EACH BOX CONTAINS AT MOST  $(\lceil \frac{n}{k} \rceil - 1)$  OBJECTS. THEN THE NUMBER OF OBJECTS  $n$  MUST SATISFY:

$$n \leq k \cdot (\lceil \frac{n}{k} \rceil - 1) < k \left( \left( \frac{n}{k} + 1 \right) - 1 \right) = k \cdot \frac{n}{k} = n,$$

WHICH SAYS  $n < n$ , A CONTRADICTION.

$\therefore$  SOME BOX MUST CONTAIN AT LEAST  $\lceil \frac{n}{k} \rceil$  OBJECTS. ///

NOTE: WE USED  $\lceil x \rceil < x + 1 \quad \forall x \in \mathbb{R}$ .

Ex.

THERE ARE 38 DIFFERENT TIME PERIODS DURING WHICH CLASSES AT A UNIVERSITY MAY BE SCHEDULED. IF THERE ARE 677 DIFFERENT CLASSES, HOW MANY ROOMS ARE NEEDED?

BY THE GPP AT LEAST ONE PERIOD MUST HAVE  $\lceil \frac{677}{38} \rceil = \lceil 17.8 \rceil = 18$  CLASSES ASSIGNED TO IT. THEREFORE AT LEAST 18 ROOMS ARE NECESSARY.

Ex.

FIND THE SMALLEST  $n$  SUCH THAT IN ANY GROUP OF  $n$  PEOPLE, AT LEAST 100 HAVE THE SAME BIRTHDAY.

WE SEEK THE SMALLEST  $n$  SUCH THAT

$$\lceil \frac{n}{366} \rceil = 100$$

$$\therefore 99 < \frac{n}{366} = 100$$

$$366 \cdot 99 < n \leq 366 \cdot 100$$

$$\therefore n = 366 \cdot 99 + 1 = 36,235$$

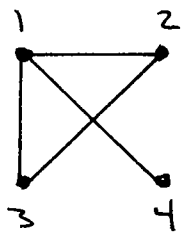
THEOREM

IN ANY GROUP OF 6 PEOPLE IN WHICH EACH PAIR ARE EITHER FRIENDS OR ENEMIES, THERE MUST EXIST EITHER A GROUP OF 3 MUTUAL FRIENDS OR A GROUP OF 3 MUTUAL ENEMIES.

IT IS HELPFUL TO STATE THIS THEOREM IN TERMS OF GRAPH THEORY.

A GRAPH  $G$  CONSISTS OF A PAIR  $(V, E)$  OF SETS CALLED VERTICES AND EDGES RESPECTIVELY. WE REQUIRE  $V \neq \emptyset$ . EACH EDGE  $e \in E$  IS AN UNORDERED PAIR OF VERTICES:  $e = \{x, y\} \subseteq V$ . WE WRITE  $e = xy$ .

Ex.

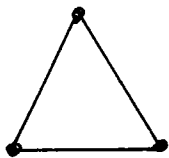
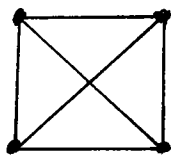
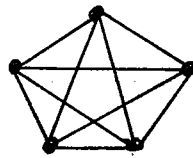
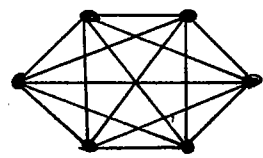


$$V = \{1, 2, 3, 4\}$$

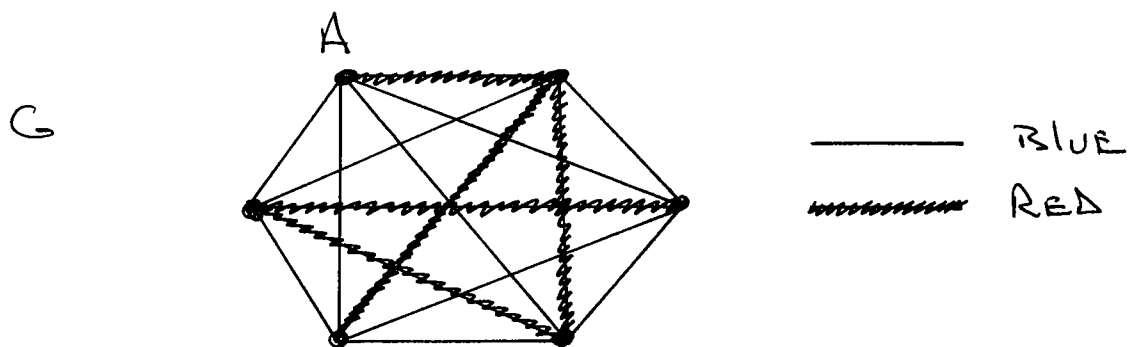
$$E = \{12, 13, 14, 23, 24, 34\}$$

WE SAY THAT VERTICES 1 AND 2 ARE ADJACENT, THAT EDGE 12 JOINS 1 TO 2, AND THAT EDGE 12 IS INCIDENT WITH VERTEX 1.

THE COMPLETE GRAPH ON  $n$  VERTICES HAS AN EDGE JOINING EACH PAIR OF VERTICES

 $n=3$  $n=4$  $n=5$  $n=6$ 

THE PRECEDING THEOREM CAN NOW BE STATED AS FOLLOWS: LET  $G$  BE THE COMPLETE GRAPH ON 6 VERTICES, AND SUPPOSE EACH EDGE OF  $G$  IS COLORED EITHER BLUE (FRIENDSHIP) OR RED (ENMITY). THEN  $G$  MUST CONTAIN EITHER A BLUE TRIANGLE OR A RED TRIANGLE.



PROOF OF THM.

LET  $A$  BE A PERSON IN THE GROUP.  
THE REMAINING 5 PEOPLE FALL INTO  
ONE OF 2 CLASSES:  $F = \{\text{FRIENDS OF } A\}$ ,  
 $E = \{\text{ENEMIES OF } A\}$ .

By the P.P. AT LEAST  $\lceil \frac{5}{2} \rceil = 3$  PEOPLE  
ARE IN THE SAME CLASS.

CASE 1 :  $|F| \geq 3$

SAY  $B, C, D$  ARE FRIENDS OF  $A$ . IF  
ANY TWO OF THESE, SAY  $B, C$ , ARE  
FRIENDS (OF EACH OTHER) THEN  $A, B, C$   
FORMS A GROUP OF 3 MUTUAL FRIENDS.  
OTHERWISE  $B, C, D$  FORM A GROUP OF  
3 MUTUAL ENEMIES.

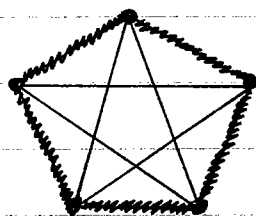
CASE 2 :  $|E| \geq 3$

FOLLOWS FROM CASE 1 BY SWAPPING 'FRIEND'  $\leftrightarrow$  'ENEMY'

IN BOTH CASES THERE IS EITHER A GROUP OF  
3 MUTUAL FRIENDS OR 3 MUTUAL ENEMIES. ///

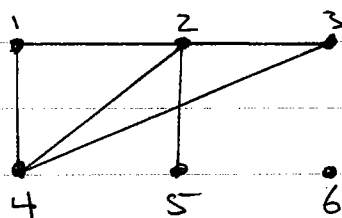
NOTE: 6 is the smallest number of people for which this conclusion is true. For example, it is possible that in a group of 5 people (in which each pair are either friends or enemies), there exists neither 3 mutual friends nor 3 mutual enemies.

EX



DEFN: Let  $G = (V, E)$  and  $x \in V$ . The degree of  $x$  is the number of edges incident with  $x$ .

EX



$$\begin{aligned} \deg(1) &= 2 & \deg(2) &= 4 \\ \deg(4) &= 3 & \deg(6) &= 0 \end{aligned}$$

THEOREM

Let  $G$  be a graph on  $n \geq 2$  vertices. Then  $G$  contains 2 vertices of like degree.

PROOF: EXERCISE

Consider subgraph  $(V - V_0, E)$  where  $V_0 = \{\text{vertices of degree } 0\}$ , then apply the Pigeonhole Principle.