

4.1 BASIC COUNTING

A STRING IS SIMPLY A FINITE SEQUENCE OF SYMBOLS FROM SOME ALPHABET.

Ex How many strings of length 2 from the alphabet $\{a, \dots, z\}$ of lower case latin letters are there?

$$\begin{array}{l}
 \overbrace{\hspace{10em}}^{26} \\
 \left. \begin{array}{l}
 aa \quad ab \quad \dots \quad az \\
 ba \quad bb \quad \dots \quad bz \\
 \vdots \quad \vdots \quad \quad \quad \vdots \\
 za \quad za \quad \dots \quad zz
 \end{array} \right\} 26 \quad 26 \cdot 26 = 676
 \end{array}$$

Product Rule

SUPPOSE A TASK T CAN BE BROKEN DOWN INTO TWO SUBTASKS T_1, T_2 TO BE PERFORMED IN SUCCESSION. SUPPOSE T_1 CAN BE PERFORMED IN n_1 WAYS, AND WHEN T_1 IS COMPLETE, T_2 CAN BE PERFORMED IN n_2 WAYS. THEN T CAN BE PERFORMED IN $n_1 \cdot n_2$ WAYS.

MORE GENERALLY SUPPOSE T CAN BE BROKEN DOWN INTO m SUCCESSIVE SUBTASKS T_1, \dots, T_m , AND THAT AFTER T_1, \dots, T_{i-1} ARE COMPLETE T_i CAN BE PERFORMED IN n_i WAYS ($1 \leq i \leq m$). THEN T CAN BE PERFORMED IN $n_1 \cdot n_2 \cdot \dots \cdot n_m$ WAYS.

Ex How many strings of length 5 from $\{a, \dots, z\}$ are there?

$$\frac{26}{1} \frac{26}{2} \frac{26}{3} \frac{26}{4} \frac{26}{5}$$

$$26^5 = 11,881,376$$

Ex How many such strings have no repeated letter?

$$\frac{26}{1} \frac{25}{2} \frac{24}{3} \frac{23}{4} \frac{22}{5}$$

$$26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 = 7,893,600$$

Ex How many bit strings of length n are there? (i.e. strings from the alphabet $\{0, 1\}$.)

$$\frac{2}{1} \frac{2}{2} \frac{2}{3} \dots \frac{2}{n}$$

$$2 \cdot 2 \cdot 2 \dots 2 = 2^n$$

Ex The number of strings of length n from an alphabet of size m is m^n .

$$\frac{m}{1} \frac{m}{2} \frac{m}{3} \dots \frac{m}{n}$$

EX. DETERMINE THE NUMBER OF STRINGS OF LENGTH n FROM AN ALPHABET OF SIZE m , IN WHICH NO SYMBOL IS REPEATED.

$$\frac{m}{1} \frac{m-1}{2} \frac{m-2}{3} \dots \frac{m-n+1}{m-n+1} = \frac{m!}{(m-n)!}$$

THEOREM

IF A, B ARE FINITE SETS, THEN SO IS $A \times B$, AND $|A \times B| = |A| \cdot |B|$

PROOF

TO FORM AN ORDERED PAIR $(x, y) \in A \times B$ ONE FIRST CHOOSES $x \in A$ IN ONE OF $|A|$ WAYS, THEN CHOOSES $y \in B$ IN ANY OF $|B|$ WAYS. //.

THEOREM

IF S IS A FINITE SET THEN SO IS $\mathcal{P}(S)$ AND $|\mathcal{P}(S)| = 2^{|S|}$.

PROOF:

THE TASK OF CONSTRUCTING A SUBSET $A \subseteq S$ BREAKS NATURALLY INTO $|S|$ SUBTASKS: FOR EACH $x \in S$, DECIDE WHETHER OR NOT $x \in A$. SINCE EACH SUBTASK CAN BE

PERFORMED in 2 ways ($x \in A$ or $x \notin A$),
THERE ARE

$$\underbrace{2 \cdot 2 \cdot 2 \cdots 2}_{|S|} = 2^{|S|}$$

WAYS TO CONSTRUCT $A \subseteq S$. THUS THERE
ARE $2^{|S|}$ SUBSETS OF S . ///

REMARK: SUPPOSE $|S| = n$, SAY S
 $= \{x_1, x_2, \dots, x_n\}$. THEN THERE IS
A BIRJECTION.

$$f: \mathcal{P}(S) \rightarrow \{\text{BIT STRINGS OF LENGTH } n\}$$

DEFINED AS FOLLOWS: FOR $A \subseteq S$, LET
 $f(A)$ BE THE BIT STRING WHOSE i 'TH
BIT IS 1 IFF $x_i \in A$.

EX. $S = \{1, 2, 3\}$, $\mathcal{P}(S) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, S\}$

$f(\emptyset) = 000$	$f(\{1, 2\}) = 110$
$f(\{1\}) = 100$	$f(\{1, 3\}) = 101$
$f(\{2\}) = 010$	$f(\{2, 3\}) = 011$
$f(\{3\}) = 001$	$f(S) = 111$

DEFIN

LET A, B BE SETS. WE DENOTE BY B^A THE SET OF FUNCTIONS WITH DOMAIN A AND CODOMAIN B .

$$B^A = \{ f : A \rightarrow B \}$$

THEOREM

IF A, B ARE FINITE SETS, THEN SO IS B^A , AND

$$|B^A| = |B|^{|A|}$$

PROOF:

SUPPOSE $|A| = m$ AND $|B| = n$. THE TASK OF CONSTRUCTING A FUNCTION $f : A \rightarrow B$ BREAKS INTO m SUCCESSIVE SUBTASKS: FOR EACH $x \in A$ CHOOSE ONE OF THE n ELEMENTS IN B AS $f(x)$. SINCE THERE ARE m SUBTASKS, EACH WITH ALTERNATIVES, THE PRODUCT RULE SAYS THERE ARE

$$\underbrace{n \cdot n \cdot n \cdots n}_m = n^m$$

WAYS TO CONSTRUCT SUCH A FUNCTION.

HENCE $|B^A| = n^m$ AS REQUIRED. ///

EX. HOW MANY FUNCTIONS $f: A \rightarrow B$ ARE INJECTIVE?

LET $|A| = m$, $|B| = n$ AS BEFORE, AND SUPPOSE $A = \{x_1, \dots, x_m\}$. NOTE THAT IF $n < m$ THERE ARE NO INJECTIVE FUNCTIONS, IF $n \geq m$ THE TASK OF CONSTRUCTING AN INJECTIVE FUNCTION $f: A \rightarrow B$ BREAKS INTO m SUBTASKS:

- choose $f(x_1) \in B$: n ways
- choose $f(x_2) \in B$: $(n-1)$ ways
- choose $f(x_3) \in B$: $(n-2)$ ways
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- choose $f(x_m) \in B$: $(n-m+1)$ ways

THUS THE NUMBER OF INJECTIVE FUNCTIONS IN B^A IN THIS CASE IS:

$$n \cdot (n-1) \cdot (n-2) \cdots (n-m+1) = \frac{n!}{(n-m)!}$$

IN GENERAL THE NUMBER OF INJECTIVE FUNCTIONS FROM A TO B IS:

$$\begin{cases} 0 & \text{if } n < m \\ \frac{n!}{(n-m)!} & \text{if } n \geq m \end{cases}$$

SUM RULE

SUPPOSE A TASK T CAN BE PERFORMED BY ONE OF TWO SUBTASKS T_1 OR T_2 , BUT NOT BOTH. SUPPOSE T_1 CAN BE PERFORMED IN n_1 WAYS, AND T_2 IN n_2 WAYS. THEN T CAN BE PERFORMED IN $n_1 + n_2$ WAYS.

STATED IN TERMS OF SETS, THE SUM RULE SAYS THAT IF A, B ARE FINITE SETS WITH $A \cap B = \emptyset$, THEN

$$|A \cup B| = |A| + |B|.$$

MORE GENERALLY IF T CAN BE PERFORMED BY DOING EXACTLY ONE OF THE SUBTASKS T_1, \dots, T_m , AND T_i CAN BE DONE IN n_i WAYS ($1 \leq i \leq m$), THEN T CAN BE PERFORMED IN $n_1 + n_2 + \dots + n_m$ WAYS.

IN TERMS OF SETS WE WOULD SAY THAT IF A_1, \dots, A_m ARE PAIRWISE DISJOINT (i.e. $A_i \cap A_j = \emptyset$ FOR $i \neq j$) THEN

$$|A_1 \cup A_2 \cup \dots \cup A_m| = |A_1| + |A_2| + \dots + |A_m|.$$

Ex. How many strings of length at most 5 can be formed from an alphabet of size 12? (include the empty string.)

we have 6 subtasks:

- Form all strings of length 5: 12^5 ways
- " " " " " 4: 12^4 ways
- " " " " " 3: 12^3 ways
- " " " " " 2: 12^2 ways
- " " " " " 1: 12^1 ways
- " " " " " 0: 12^0 ways.

By the sum rule there are

$$12^0 + 12^1 + 12^2 + 12^3 + 12^4 + 12^5 = \frac{12^6 - 1}{12 - 1} = 271,453$$

strings of length at most 5.

Ex. A computer system uses passwords consisting of 6 to 8 characters which are either upper or lower case alphabetic (52) or digits (10) subject to the following constraints: A password must contain at least 1 digit and at least 1 alphabetic character. How many valid passwords are there?

LET P BE THE NUMBER OF SUCH
 PASSWORDS, AND P_i BE THE NUMBER
 OF PASSWORDS OF LENGTH i ($i=6,7,8$).
 THEN $P = P_6 + P_7 + P_8$ BY THE SUM
 RULE, AND

$$P_6 = 62^6 - 52^6 - 10^6 \approx (3.7) \cdot 10^{10}$$

$$P_7 = 62^7 - 52^7 - 10^7 \approx (1.0) \cdot 10^{12}$$

$$P_8 = 62^8 - 52^8 - 10^8 \approx (1.6) \cdot 10^{14}$$

$$\therefore P \approx (1.61037) \cdot 10^{14} \approx 161 \text{ trillion}$$

PRINCIPLE OF INCLUSION-EXCLUSION (PIE)

RECALL THAT IF A, B ARE FINITE
 SETS, NOT NECESSARILY DISJOINT, THEN

$$|A \cup B| = |A| + |B| - |A \cap B|$$

EX. HOW MANY BIT STRINGS OF LENGTH 7
 EITHER BEGIN WITH 2 0'S OR END
 WITH 3 1'S?

LET $A = \{00xxxxx\}$, $B = \{xxxx111\}$. THEN
 $A \cap B = \{00xx111\}$ AND

$$|A \cup B| = 2^5 + 2^4 - 2^2 = 44$$

RECALL THE EULER TOTIENT FUNCTION
 $\phi(n)$ IS DEFINED TO BE THE NUMBER
 OF NUMBERS IN $\{1, 2, \dots, n\}$ WHICH ARE
 RELATIVELY PRIME TO n .

EX. DETERMINE $\phi(1000)$ (NOTE THIS
 WAS AN EARLIER HW ASSIGNMENT.)

$$\phi(1000) = 1000 - (\# \text{ OF } \# \text{'S NOT REL. PRM. TO } 1000)$$

A NUMBER IS NOT RELATIVELY PRIME TO 1000
 IFF IT HAS A FACTOR IN COMMON WITH
 $1000 = 2^3 \cdot 5^3$, I.E. DIVISIBLE BY 2 OR 5.

$$\begin{aligned} (\# \text{ OF } \# \text{'S DIVISIBLE BY } 2) &= \frac{1000}{2} = 500 \\ (\# \text{ OF } \# \text{'S DIVISIBLE BY } 5) &= \frac{1000}{5} = 200 \\ (\# \text{ OF } \# \text{'S DIVISIBLE BY BOTH}) &= \frac{1000}{2 \cdot 5} = \frac{1000}{10} = 100 \end{aligned}$$

$$\therefore \phi(1000) = 1000 - (500 + 200 - 100) = 400$$

GENERALIZATIONS OF PIE:

$$\begin{aligned} |A_1 \cup A_2 \cup A_3| &= (|A_1| + |A_2| + |A_3|) \\ &\quad - (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|) \\ &\quad + |A_1 \cap A_2 \cap A_3| \end{aligned}$$

$$|\bigcup_{i=1}^4 A_i| = \sum_i |A_i| - \sum_{i < j} |A_i \cap A_j| + \sum_{i < j < k} |A_i \cap A_j \cap A_k| - |A_1 \cap A_2 \cap A_3 \cap A_4|$$

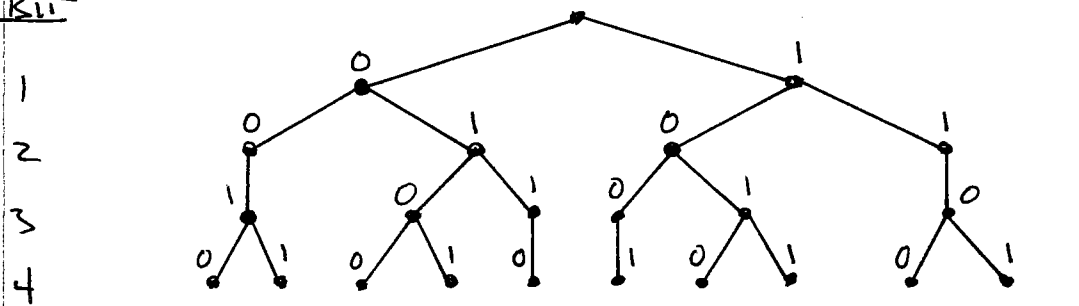
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$$|\bigcup_{i=1}^n A_i| = \sum_i |A_i| - \sum_{i < j} |A_i \cap A_j| + \sum_{i < j < k} |A_i \cap A_j \cap A_k| - \sum_{i < j < k < l} |A_i \cap A_j \cap A_k \cap A_l| + \dots + (-1)^{n+1} |A_1 \cap A_2 \cap \dots \cap A_n|$$

TREE DIAGRAMS

EX. HOW MANY BIT STRINGS OF LENGTH 4 CONTAIN NEITHER 3 CONSECUTIVE 0'S NOR 3 CONSECUTIVE 1'S ?

BIT



STRINGS: { 001, 010, 011, 100, 101, 110, 111, 000, 111 }

∴ THERE ARE 10 SUCH STRINGS.