

### 3.4 RECURSIVE DEFINITIONS

EX DEFINE  $f: \mathbb{N} \rightarrow \mathbb{Z}$  BY

$$\begin{cases} f(0) = 3 \\ f(n) = 3f(n-1) + 2 \quad \forall n \geq 1 \end{cases}$$

THEN  $f(0) = 3, f(1) = 11, f(2) = 35, \dots$

A RECURSIVE DEFINITION OF  $f: \mathbb{N} \rightarrow \mathcal{S}$   
SPECIFIER

- INITIAL CONDITION(S) :  $f(0)$
- RECURSIVE FORMULA : A RULE FOR COMPUTING  $f(n)$  IN TERMS OF  $f(n-1), f(n-2), \dots, f(0)$ .

SOME WELL KNOWN RECURSIVE FUNCTIONS.

EX

$$\left. \begin{cases} g(0) = 1 \\ g(n) = n \cdot g(n-1) \quad \forall n \geq 1 \end{cases} \right\} \rightarrow g(n) = n!$$

EX LET  $b \in \mathbb{R}$  BE A FIXED CONSTANT.

$$\left. \begin{cases} h(0) = 1 \\ h(n) = b \cdot h(n-1) \quad \forall n \geq 1 \end{cases} \right\} \rightarrow h(n) = b^n.$$

SUCH A FUNCTION (i.e. SEQUENCE) NEEDS NOT BEGIN AT  $n=0$ .

EX. DEFINE  $S: \mathbb{Z}^+ \rightarrow \mathbb{Z}$  BY

$$\left. \begin{array}{l} S(1) = 1 \\ S(n) = S(n-1) + n \quad \forall n \geq 2 \end{array} \right\} \rightarrow S(n) = \sum_{k=1}^n k = \frac{n(n+1)}{2}$$

THERE MAY BE MORE THAN ONE INITIAL CONDITION. IN GENERAL IF  $f(n)$  DEPENDS ON  $f(n-k)$  THERE MUST BE AT LEAST  $k$  INITIAL TERMS GIVEN.

EX. FIBONACCI SEQUENCE

$$\left\{ \begin{array}{l} F_0 = 0 \\ F_1 = 1 \\ F_n = F_{n-1} + F_{n-2} \quad \forall n \geq 2 \end{array} \right.$$

$$\begin{array}{ll} \therefore F_0 = 0 & F_7 = 13 \\ F_1 = 1 & F_8 = 21 \\ F_2 = 1 & F_9 = 34 \\ F_3 = 2 & F_{10} = 55 \\ F_4 = 3 & \vdots \\ F_5 = 5 & \\ F_6 = 8 & \end{array}$$

THEOREMS INVOLVING RECURSIVELY DEFINED FUNCTIONS ARE OFTEN PROVED BY INDUCTION.

Ex Show  $\forall n \geq 1$

$$F_1 + F_3 + F_5 + \dots + F_{2n-1} = F_{2n}$$

i.e.  $\sum_{k=1}^n F_{2k-1} = F_{2n}$

PROOF:

I.  $P(1)$  says  $F_1 = F_{2 \cdot 1}$ , i.e.  $1 = 1$ , which is true

II. Let  $n \geq 1$  and assume  $\sum_{k=1}^n F_{2k-1} = F_{2n}$ .

Then

$$\sum_{k=1}^{n+1} F_{2k-1} = \left( \sum_{k=1}^n F_{2k-1} \right) + F_{2(n+1)-1}$$

$$= F_{2n} + F_{2n+1} \quad (\text{By ind. Hyp.})$$

$$= F_{2n+2}$$

$$= F_{2(n+1)}.$$

$\therefore$  Result follows by PMI. ///

Ex For all  $n \geq 1$

$$F_0 F_1 + F_1 F_2 + F_2 F_3 + \dots + F_{n-1} F_n = \begin{cases} F_n^2 & (n \text{ even}) \\ F_{n-1} F_{n+1} & (n \text{ odd}) \end{cases}$$

PROOF: LET  $P(n)$  BE THE FORMULA

$$\sum_{k=1}^n F_{k-1} F_k = \begin{cases} F_n^2 & (n \text{ even}) \\ F_{n-1} F_{n+1} & (n \text{ odd}) \end{cases}$$

I.  $n=1$  is odd so  $P(1)$  says  $F_0 F_1 = F_0 F_2$ ,  
i.e.  $0 \cdot 1 = 0 \cdot 1$ , i.e.  $0 = 0$ .

II. LET  $n \geq 1$  AND ASSUME  $P(n)$  HOLDS.

CASE 1:  $n$  even  $\rightarrow n+1$  odd

$$\text{ASSUME } P(n): \sum_{k=1}^n F_{k-1} F_k = F_n^2$$

$$\text{SHOW } P(n+1): \sum_{k=1}^{n+1} F_{k-1} F_k = F_{(n+1)-1} F_{(n+1)+1}$$

$$\sum_{k=1}^{n+1} F_{k-1} F_k = \left( \sum_{k=1}^n F_{k-1} F_k \right) + F_n F_{n+1}$$

$$= F_n^2 + F_n F_{n+1} \quad (\text{by ind. Hyp.})$$

$$= F_n (F_n + F_{n+1})$$

$$= F_n F_{n+2} \quad (\text{By FIB. RECURRENCE.})$$

$\therefore P(n+1)$  is TRUE.

CASE 2:  $n$  ODD  $\rightarrow n+1$  even

$$\text{Assume } P(n): \sum_{k=1}^n F_{k-1} F_k = F_{n-1} F_{n+1}$$

$$\text{Show } P(n+1): \sum_{k=1}^{n+1} F_{k-1} F_k = F_{n+1}^2$$

$$\sum_{k=1}^{n+1} F_{k-1} F_k = \left( \sum_{k=1}^n F_{k-1} F_k \right) + F_{(n+1)-1} F_{n+1}$$

$$= F_{n-1} F_{n+1} + F_n F_{n+1} \quad (\text{By IND. Hyp.})$$

$$= (F_{n-1} + F_n) F_{n+1}$$

$$= F_{n+1} \cdot F_{n+1} \quad (\text{By FIB. RECURRENCE.})$$

$$= F_{n+1}^2$$

$\therefore P(n+1)$  is TRUE.

$\therefore$  in EITHER CASE  $P(n) \rightarrow P(n+1)$ .

$\therefore$  RESULT follows from PMI.

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READ LAMBE'S THEOREM P. 260.

### RECURSIVELY DEFINED SETS

TO DEFINE A SET  $S$  RECURSIVELY, WE SPECIFY:

- (1) AN INITIAL SUBSET  $S_0$  OF  $S$
- (2) A RULE (OR RULES) WHICH SHOW HOW TO CONSTRUCT NEW ELEMENTS OF  $S$  FROM OLD ONES.

THE SET  $S$  THEN CONSISTS OF EXACTLY THOSE ELEMENTS WHICH ARE CONSTRUCTED BY (1) OR (2)

EX. DEFINE  $S \subseteq \mathbb{Z}^+$  BY

- (1)  $5 \in S$
- (2) IF  $x \in S$  AND  $y \in S$ , THEN  $x+y \in S$

$S$  CERTAINLY CONTAINS  $5, 10, 15, 20, 25, \dots$

WHAT EXACTLY IS THE SET  $S$ ?

LET  $A = \{sn \mid n \in \mathbb{Z}^+\}$ , i.e.  $A$  IS THE SET OF POSITIVE MULTIPLES OF  $s$ .

WE WILL SHOW THAT  $S = A$ .

PROOF OF  $A \subseteq S$

LET  $P(n) = 'sn \in S'$ .

I.  $P(1)$  SAYS  $s \in S$ , WHICH IS TRUE BY (1).

II. LET  $n \geq 1$  AND ASSUME  $P(n)$  IS TRUE, i.e. FOR THIS  $n$ ,  $sn \in S$ . THEN  $s(n+1) = sn + s \in S$  BY RULE (2).  $\therefore P(n+1)$  IS TRUE.

$\therefore \forall n : sn \in S$  BY PMI  
 $\therefore A \subseteq S$ .

PROOF OF  $S \subseteq A$

OBSERVE THAT  $A$  SATISFIES RULES (1) AND (2):

(1)  $s \in A$

(2)  $x \in A \wedge y \in A \rightarrow x + y \in A$

(SINCE  $sn + sm = s(m+n) \in A$ .)

THUS ANY NUMBER WHICH CAN BE  
CONSTRUCTED BY RULES (1) & (2) MUST  
BELONG TO  $A$ . BUT  $S$  IS JUST  
THE SET OF ALL SUCH NUMBERS.

$$\therefore S \subseteq A.$$

$A \subseteq S$  AND  $S \subseteq A$  IMPLIES  $S = A$ , AS  
REQUIRED.

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READ: RECURSIVE DEFINITIONS OF STRINGS,  
WELL FORMED FORMULAS, ROOTED TREES,  
BINARY TREES, AND STRUCTURAL INDUCTION.  
P. 262-270