4.4 Binomial Coefficients

Recall $C(n,k) = \binom{n}{k} = \dfrac{n!}{k!(n-k)!}$.

We will freely use both notations: $C(n,k)$ and $\binom{n}{k}$.

Observe:

$\binom{0}{0} = 1$

$\binom{1}{0} = 1$, $\binom{1}{1} = 1$

$\binom{2}{0} = 1$, $\binom{2}{1} = 2$, $\binom{2}{2} = 1$

$\binom{3}{0} = 1$, $\binom{3}{1} = 3$, $\binom{3}{2} = 3$, $\binom{3}{3} = 1$

Continuing in this manner we get Pascal's Triangle

n	
0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1
6	1 6 15 20 15 6 1
7	1 7 21 35 35 21 7 1
8	1 8 28 56 70 56 28 8 1
⋮	⋮

OBSERVE THAT EACH INTERIOR ENTRY IN THE TRIANGLE IS THE SUM OF THE NEIGHBORING ENTRIES IN THE PRECEDING ROW.

THEOREM (PASCAL'S IDENTITY)
LET $1 \le k \le n$. THEN

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

PROOF: (ALGEBRAIC)

$$\binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(k-1)!(n-(k-1))!} + \frac{n!}{k!(n-k)!}$$

$$= \frac{n!\,k}{k!(n-k+1)!} + \frac{n!\,(n-k+1)}{k!(n-k+1)!}$$

$$= \frac{n!\,(\cancel{k}+n-\cancel{k}+1)}{k!\,(n-k+1)!}$$

$$= \frac{n!\,(n+1)}{k!\,(n-k+1)!}$$

$$= \frac{(n+1)!}{k!\,((n+1)-k)!}$$

$$= \binom{n+1}{k}. \qquad ///.$$

PROOF: (COMBINATORIAL)
LET $|T| = n+1$, $x \in T$, AND $S = T - \{x\}$.
THEN $|S| = n$.

A K-ELEMENT SUBSET OF T CAN BE CONSTRUCTED BY PERFORMING EITHER OF THE FOLLOWING (MUTUALLY EXCLUSIVE) SUBTASKS:

- INCLUDE x: CHOOSE A $(k-1)$ ELEMENT SUBSET OF S AND COMBINE IT WITH x. THIS CAN BE DONE IN $\binom{n}{k-1}$ WAYS.

- DO NOT INCLUDE x: CHOOSE A k-ELEMENT SUBSET OF S. THIS CAN BE DONE IN $\binom{n}{k}$ WAYS.

BY THE SUM RULE THE NUMBER OF K ELEMENT SUBSETS OF T IS $\binom{n}{k-1} + \binom{n}{k}$, WHENCE

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}. \qquad ///$$

THERE ARE MANY CASES IN WHICH THE COMBINATORIAL PROOF IS SIMPLE AND STRAIGHTFORWARD, WHILE THE ALGEBRAIC PROOF IS EXCESSIVELY COMPLEX.

THEOREM

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

FOR ALL $n \geq 0$.

PROOF: (COMBINATORIAL)
BOTH SIDES OF THE FORMULA COUNT THE NUMBER OF SUBSETS OF AN n ELEMENT SET. ///

THE BINOMIAL THEOREM

OBSERVE THAT

$$
\begin{aligned}
(x+y)^0 &= 1 \\
(x+y)^1 &= x+y \\
(x+y)^2 &= x^2 + 2xy + y^2 \\
(x+y)^3 &= x^3 + 3x^2y + 3xy^2 + y^3 \\
(x+y)^4 &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 \\
(x+y)^5 &= x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5 \\
&\vdots
\end{aligned}
$$

NOTICE THAT EACH TERM IN THE EXPANSION OF $(x+y)^n$ IS OF THE FORM $x^{n-k}y^k$ $(0 \leq k \leq n)$ TIMES SOME COEFFICIENT. NOTE ALSO THAT THE COEFFICIENTS ARE FROM PASCAL'S TRIANGLE.

THEOREM

LET $n \in \mathbb{N}$ AND $x, y \in \mathbb{R}$. THEN

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k .$$

PROOF: (COMBINATORIAL)

WHEN

$$(*) \qquad (x+y)^n = \underset{1}{(x+y)}\underset{2}{(x+y)} \cdots\cdots \underset{n}{(x+y)}$$

IS FULLY EXPANDED (AND BEFORE ANY LIKE TERMS ARE COMBINED) WE HAVE 2^n TERMS, EACH OF THE FORM $x^{n-k} y^k$ FOR SOME $0 \le k \le n$.

FIX A PARTICULAR k IN THIS RANGE. TO DETERMINE THE COEFFICIENT OF $x^{n-k} y^k$ (AFTER LIKE TERMS ARE COMBINED) WE MUST COUNT THE NUMBER OF WAYS THIS TERM CAN BE OBTAINED IN THE EXPANSION.

TO OBTAIN $x^{n-k} y^k$ WE MUST SELECT x FROM $(n-k)$ OF THE n FACTORS IN $(*)$ AND SELECT y FROM THE REMAINING k FACTORS.

TO CHOOSE WHICH k OF THE n FACTORS WILL CONTRIBUTE y IS ESSENTIALLY TO CHOOSE A k ELEMENT SUBSET OF THE SET $\{1, 2, \ldots, n\}$. THIS CAN BE DONE IN $\binom{n}{k}$ WAYS.

HENCE $x^{n-k}y^k$ OCCURS $\binom{n}{k}$ TIMES IN THE FULL EXPANSION. AFTER COMBINING LIKE TERMS, THE COEFFICIENT OF $x^{n-k}y^k$ IS $\binom{n}{k}$, AS REQUIRED.

///

<u>Corollary</u>

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

<u>PROOF</u>: LET $x=y=1$ IN BINOMIAL THM. ///

<u>Corollary</u>

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$$

<u>PROOF</u>: LET $x=1$, $y=-1$ IN BINOMIAL THM. ///.

WE GIVE A SECOND (ALGEBRAIC) PROOF OF THE BINOMIAL THEOREM.

THIS PROOF PROCEEDS BY INDUCTION AND USES PASCAL'S IDENTITY.

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

PROOF: (ALGEBRAIC)
LET $P(n)$ DENOTE THE FORMULA:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

I. $P(0)$ is $(x+y)^0 = \binom{0}{0} x^0 y^0$, WHICH SAYS $1=1$.

II. LET $n \geq 0$ AND ASSUME, FOR THIS n THAT $P(n)$ IS TRUE:

ASSUME: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$

WE MUST SHOW $P(n+1)$ IS TRUE:

SHOW: $(x+y)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} x^{(n+1)-k} y^k$.

OBSERVE

$$\begin{aligned}(x+y)^{n+1} &= (x+y)(x+y)^n \\ &= (x+y) \cdot \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \quad \left(\text{BY INDUCTION HYPOTHESIS}\right)\end{aligned}$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^{n-k+1} y^{k} + \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^{k+1}$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^{n-k+1} y^{k} + \sum_{k=1}^{n+1} \binom{n}{k-1} x^{n-(k-1)} y^{(k-1)+1}$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^{n+1-k} y^{k} + \sum_{k=1}^{n+1} \binom{n}{k-1} x^{n+1-k} y^{k}$$

$$= \binom{n}{0} x^{n+1} y^{0} + \left[\sum_{k=1}^{n} \binom{n}{k} x^{n+1-k} y^{k} + \sum_{k=1}^{n} \binom{n}{k-1} x^{n+1-k} y^{k} \right] + \binom{n}{n} x^{0} y^{n+1}$$

$$= \binom{n+1}{0} x^{n+1} y^{0} + \sum_{k=1}^{n} \left[\binom{n}{k} + \binom{n}{k-1} \right] x^{n+1-k} y^{k} + \binom{n+1}{n+1} x^{0} y^{n+1}$$

$$= \binom{n+1}{0} x^{n+1} y^{0} + \sum_{k=1}^{n} \binom{n+1}{k} x^{n+1-k} y^{k} + \binom{n+1}{n+1} x^{0} y^{n+1}$$

$$= \sum_{k=0}^{n+1} \binom{n+1}{k} x^{(n+1)-k} y^{k} .$$

$\therefore P(n+1)$ Holds.

$\therefore \forall n\, P(n)$ BY THE PRINCIPLE OF MATHEMATICAL INDUCTION.

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THEOREM (VANDERMONDE'S IDENTITY)

$$\binom{m+n}{k} = \sum_{j=0}^{k} \binom{m}{k-j}\binom{n}{j}$$

FOR $0 \le k \le \min(m,n)$.

PROOF: (COMBINATORIAL)
LET A, B BE TWO SETS WHICH SATISFY $|A| = m$, $|B| = n$, AND $A \cap B = \emptyset$. THEN $|A \cup B| = m+n$.

THE LHS OF THE ABOVE FORMULA COUNTS THE NUMBER OF k ELEMENT SUBSETS OF $A \cup B$.

SUCH A SUBSET CAN BE CONSTRUCTED AS FOLLOWS:

PICK j IN THE RANGE $0 \le j \le k$ THEN:

- CHOOSE j ELEMENTS FROM B: $\binom{n}{j}$ WAYS,
- CHOOSE $k-j$ ELEMENTS FROM A: $\binom{m}{k-j}$ WAYS,

THEN COMBINE ALL $j + (k-j) = k$ ELEMENTS TO FORM A k ELEMENT SUBSET OF $A \cup B$.

By the product rule the two sub-tasks can be performed in

$$\binom{m}{k-j}\binom{n}{j}$$

ways. Doing this for each $j=0,1,\dots,k$ gives all k element subsets of $A\cup B$. By the sum rule the number of such subsets is

$$\sum_{j=0}^{k}\binom{m}{k-j}\cdot\binom{n}{j}.$$

Hence LHS and RHS of the above formula count the very same thing, and are therefore equal.

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<u>Corollary</u> $\displaystyle\binom{2n}{n}=\sum_{j=0}^{n}\binom{n}{j}^2$

for any $n\geq 0$.

<u>Proof</u>: Let $n=m=k$ in Vandermonde's identity.

$$\binom{2n}{n}=\sum_{j=0}^{n}\binom{n}{n-j}\binom{n}{j}=\sum_{j=0}^{n}\binom{n}{j}^2$$

since $\binom{n}{n-j}=\binom{n}{j}$ by an earlier theorem.

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