

EX $\forall x P(x) = 'x^2 < 0'$ (UNIVERSE \mathbb{R})

$\exists x P(x) = 'THERE IS AT LEAST ONE REAL NUMBER WHOSE SQUARE IS NEGATIVE'$

WHAT IS THE TRUTH VALUE OF THIS PROPOSITION?
WHAT HAPPENS IF WE REPLACE THE UNIVERSE \mathbb{R} BY \mathbb{Q} ?

CONSIDER THE MEANING OF QUANTIFICATION IN A FINITE UNIVERSE, SAY $U = \{a, b\}$.

$$\forall x P(x) \equiv P(a) \wedge P(b)$$

$$\exists x P(x) \equiv P(a) \vee P(b)$$

THEN

$$\neg \forall x P(x) \equiv \neg (P(a) \wedge P(b))$$

$$\equiv \neg P(a) \vee \neg P(b)$$

$$\equiv \exists x \neg P(x)$$

SIMILARLY $\neg \exists x P(x) \equiv \forall x \neg P(x)$.

THESE IDENTITIES ALSO HOLD FOR INFINITE UNIVERSES.

STATEMENT	TRUE WHEN	FALSE WHEN
$\forall x P(x)$	$P(x)$ TRUE FOR ALL x	$P(x)$ FALSE FOR AT LEAST ONE x
$\exists x P(x)$	$P(x)$ TRUE FOR AT LEAST ONE x	$P(x)$ FALSE FOR ALL x

A VARIABLE x IN A PROPOSITIONAL FUNCTION IS SAID TO BE BOUND IF IT HAS EITHER BEEN QUANTIFIED, OR HAS BEEN ASSIGNED A VALUE. OTHERWISE x IS CALLED FREE.

— Ex. $P(x, y) = 'x + y = y + x'$
 $Q(y, z) = 'y - z = z - y'$

- $\forall x \forall y P(x, y)$ (x, y BOUND)
- $\forall x P(x, y)$ (x BOUND, y FREE)
- $P(x, y)$ (x, y FREE)
- $\forall x \exists y P(x, y) \vee Q(y, z)$ (z FREE)
- $\exists z \exists x P(x, z) \wedge Q(z, z)$ (x, y, z BOUND)
- $\forall z Q(y, z)$ (z BOUND, y FREE)

ANY EXPRESSION WHICH CONTAINS FREE VARIABLES IS NOT A PROPOSITION SINCE ITS TRUTH VALUE DEPENDS ON THE VALUES OF THOSE VARIABLES.

CONSIDER THE SENTENCE

'ALL HORSES IN CALIFORNIA ARE BLUE'

HOW CAN WE EXPRESS THIS SYMBOLICALLY USING QUANTIFIERS?

~~EX~~ LET $U = \{ \text{HORSES IN CALIFORNIA} \}$ AND $R(x) = 'x \text{ IS BLUE}'$. THEN

$$\forall x R(x)$$

~~EX~~ LET $U = \{ \text{HORSES} \}$ AND $C(x) = 'x \text{ LIVES IN CALIFORNIA}'$. THEN

$$\forall x (C(x) \rightarrow R(x))$$

~~EX~~ LET $U = \{ \text{LIVING THINGS} \}$ AND $H(x) = 'x \text{ IS A HORSE}'$. THEN

$$\forall x (H(x) \wedge C(x) \rightarrow R(x))$$

~~EX~~ (WRONG) $\forall x (H(x) \wedge C(x) \wedge R(x))$.
WHAT DOES THIS SAY?

RED: EXAMPLES FROM LEWIS CARROLL P.37.

(1.4) NESTED QUANTIFIERS

LET $P(x, y)$ BE ANY PROPOSITIONAL FUNCTION ON ANY UNIVERSE. THEN

$$\forall x \forall y P(x, y) \equiv \forall y \forall x P(x, y)$$

$$\exists x \exists y P(x, y) \equiv \exists y \exists x P(x, y)$$

BUT $\forall x \exists y P(x, y) \neq \exists y \forall x P(x, y)$

TO ILLUSTRATE THE LAST STATEMENT LET $P(x, y) = 'x < y'$ (UNIVERSE \mathbb{R}).
THEN

$$\forall x \exists y P(x, y) \text{ is TRUE}$$

$$\exists y \forall x P(x, y) \text{ is FALSE}$$

EX. $L(x, y) = 'x \text{ loves } y'$, $U = \{\text{PEOPLE}\}$.

'EVERY BODY LOVES SOMEBODY' $\equiv \forall x \exists y L(x, y)$

'SOMEBODY LOVES EVERYBODY' $= \exists x \forall y L(x, y)$

'NOBODY LOVES EVERYBODY' =

$$\neg \exists x \forall y L(x, y) \equiv \forall x \neg \forall y L(x, y)$$

$$\equiv \forall x \exists y \neg L(x, y)$$

1. THERE IS EXACTLY ONE PERSON WHOM EVERYBODY LOVES.

$$\exists y \left(\forall x L(x, y) \wedge \forall z (z \neq y \rightarrow \exists w \neg L(w, z)) \right)$$

(1.5) METHODS OF PROOF

OUR FORMAL RULES OF INFERENCE WILL HAVE THE FOLLOWING FORM

$$\begin{array}{l} \text{HYPOTHESIS 1} \\ \text{(HYPOTHESIS 2)} \\ \hline \therefore \text{CONCLUSION} \end{array}$$

EACH RULE IS BASED ON A CORRESPONDING TAUTOLOGY OF THE FORM

$$(H_1 \wedge H_2) \rightarrow C$$

RUNNING EXAMPLE

- P = 'IT IS RAINING TODAY'
- Q = 'CLASS IS CANCELLED'
- r = 'WE GO TO THE BEACH'

ADDITION

$$\begin{array}{l} P \\ \hline \therefore P \vee Q \end{array}$$

$$P \rightarrow (P \vee Q)$$

SIMPLIFICATION

$$\begin{array}{l} P \wedge Q \\ \hline \therefore P \end{array}$$

$$(P \wedge Q) \rightarrow P$$

CONJUNCTION

$$\begin{array}{l} P \\ \underline{Q} \\ \therefore P \wedge Q \end{array} \quad (P \wedge Q) \rightarrow (P \wedge Q)$$

MODUS PONENS

$$\begin{array}{l} P \\ \underline{P \rightarrow Q} \\ \therefore Q \end{array} \quad [P \wedge (P \rightarrow Q)] \rightarrow Q$$

MODUS TOLLENS

$$\begin{array}{l} \neg Q \\ \underline{P \rightarrow Q} \\ \therefore \neg P \end{array} \quad [\neg Q \wedge (P \rightarrow Q)] \rightarrow \neg P$$

HYPOTHETICAL SYLLOGISM

$$\begin{array}{l} P \rightarrow Q \\ \underline{Q \rightarrow R} \\ \therefore P \rightarrow R \end{array} \quad [(P \rightarrow Q) \wedge (Q \rightarrow R)] \rightarrow (P \rightarrow R)$$

DISJUNCTIVE SYLLOGISM

$$\begin{array}{l} P \vee Q \\ \underline{\neg P} \\ \therefore Q \end{array} \quad [(P \vee Q) \wedge \neg P] \rightarrow Q$$

Resolution

$$\begin{array}{l} P \vee Q \\ \hline \neg P \vee R \\ \hline \therefore Q \vee R \end{array}$$

$$[(P \vee Q) \wedge (\neg P \vee R)] \rightarrow (Q \vee R)$$

A PROOF CONSISTS OF A SEQUENCE OF PROPOSITIONS BEGINNING WITH ONE OR MORE HYPOTHESES WHICH ARE ASSUMED TO BE TRUE AND ENDING WITH A CONCLUSION. EACH PROPOSITION IN THE SEQUENCE WHICH IS NOT A HYPOTHESIS MUST FOLLOW FROM PREVIOUS PROPOSITIONS USING VALID INFERENCE RULES, LOGICAL EQUIVALENCES, OR PREVIOUSLY PROVEN STATEMENTS.

EX.

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|--------------------------------|---------------|--------------|------------------------------|-------------------------|
| 1.) $\neg P \vee \neg Q$ | \rightarrow | $r \wedge s$ | } <u>HYPOTHESES</u> | <u>CONCLUSION</u> : P |
| 2.) $S \rightarrow U$ | | | | |
| 3.) $\neg U$ | | | | |
| 4.) $\neg S$ | | | MODUS TOLLENS 2,3 | |
| 5.) $\neg r \vee \neg s$ | | | ADDITION 4, COMMUTATIVE | |
| 6.) $\neg(r \wedge s)$ | | | DE MORGAN 5 | |
| 7.) $\neg(\neg P \vee \neg Q)$ | | | MODUS TOLLENS 1,6 | |
| 8.) $P \wedge Q$ | | | DE MORGAN, DOUBLE NEGATION 7 | |
| 9.) P | | | SIMPLIFICATION 8 | |

Ex.

- | | | | |
|-----|------------------------|---------------------|-------------------------|
| 1.) | $\neg P \wedge \neg Q$ | } <u>HYPOTHESES</u> | <u>CONCLUSION</u> : t |
| 2.) | $r \rightarrow p$ | | |
| 3.) | $\neg r \rightarrow s$ | | |
| 4.) | $s \rightarrow t$ | | |
| 5.) | $\neg P$ | SIMPLIFICATION 1 | |
| 6.) | $\neg r$ | MODUS TOLLENS 2, 5 | |
| 7.) | s | MODUS PONENS 3, 6 | |
| 8.) | t | MODUS PONENS 4, 7 | |

SOME FALLACIES

EX. AFFIRMING THE CONCLUSION

'IF IT RAINS TODAY, THEN CLASS IS CANCELLED.'
 'CLASS IS CANCELLED'
 \therefore 'IT IS RAINING TODAY'

$[(P \rightarrow Q) \wedge Q] \rightarrow P$ IS NOT A TAUTOLOGY.

EX. DENYING THE HYPOTHESIS

'IF IT RAINS TODAY, THEN CLASS IS CANCELLED'
 'IT IS NOT RAINING TODAY'
 \therefore 'CLASS IS NOT CANCELLED'

$[(P \rightarrow Q) \wedge \neg P] \rightarrow \neg Q$ IS NOT A TAUTOLOGY

INFERENCE RULES FOR QUANTIFIERS

UNIVERSAL INSTANTIATION

$$\underline{\forall x P(x)}$$

$$\therefore P(c)$$

ALL HORSES ARE BLUE

\therefore TRIGGER IS BLUE

UNIVERSAL GENERALIZATION

$$\underline{P(c) \text{ FOR AN ARBITRARY } c}$$

$$\therefore \forall x P(x)$$

AN ARBITRARILY SELECTED HORSE IS BLUE

\therefore ALL HORSES ARE BLUE

EXISTENTIAL INSTANTIATION

$$\underline{\exists x P(x)}$$

$$\therefore P(c) \text{ FOR SOME } c$$

THERE EXISTS A BLUE HORSE

FOR SOME HORSE c , c IS BLUE

EXISTENTIAL GENERALIZATIONS

$$\underline{P(c) \text{ FOR SOME } c}$$

$$\therefore \exists x P(x)$$

TRIGGER IS BLUE

\therefore THERE EXISTS A BLUE HORSE