3.3 MATHEMATICAL INDUCTION

EX. OBSERVE

$$1 = 1^2$$
$$1 + 3 = 2^2$$
$$1 + 3 + 5 = 3^2$$
$$1 + 3 + 5 + 7 = 4^2$$
$$\vdots$$
$$1 + 3 + 5 + \cdots + (2n-1) = n^2$$

i.e.
$$\sum_{k=1}^{n} (2k-1) = n^2$$

ANY PARTICULAR INSTANCE OF THIS FORMULA IS EASY TO VERIFY, BUT HOW CAN WE PROVE ALL INSTANCES?

LET $P(n)$ BE A PROPOSITIONAL FUNCTION WITH UNIVERSE $\mathbb{Z}^+$, i.e.

$$P : \mathbb{Z}^+ \to \{\text{false}, \text{true}\}$$

SUPPOSE WE WISH TO PROVE THE STATEMENT:

$$\forall n \, P(n)$$

i.e. $P(n) =$ 'true' FOR ALL POSITIVE INTEGERS n.

A PROOF OF $\forall n\, P(n)$ BY MATHEMATICAL INDUCTION PROCEEDS IN TWO STEPS.

I. BASE: PROVE THAT $P(1)$ IS TRUE

II. INDUCTION: PROVE $\forall n\, (P(n) \rightarrow P(n+1))$
i.e. LET $n \geq 1$, ASSUME $P(n)$ IS TRUE, AND SHOW AS A CONSEQUENCE THAT $P(n+1)$ IS TRUE.

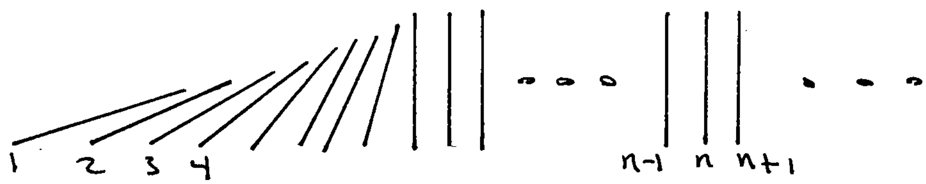
ONCE STEPS I AND II ARE COMPLETE WE MAY THEN CONCLUDE $\forall n\, P(n)$.

THE STATEMENT $P(n)$ IS CALLED THE INDUCTION HYPOTHESIS, SINCE IT IS ASSUMED TO BE TRUE IN THE INDUCTION STEP.

SOME STUDENTS BELIEVE THAT THIS ASSUMPTION CONSTITUTES CIRCULAR REASONING, BUT IN FACT WE DO NOT ASSUME $\forall n\, P(n)$. INSTEAD WE ASSUME $P(n)$ TO BE TRUE FOR ONE PARTICULAR n, THEN SHOW AS A CONSEQUENCE THAT $P(n+1)$ IS TRUE.

THUS WE SHOW $P(1)$ IS TRUE, AND SINCE $P(1) \rightarrow P(2)$, WE HAVE $P(2)$ IS TRUE. BUT SINCE $P(2) \rightarrow P(3)$, WE KNOW $P(3)$ IS TRUE, AND SO ON.

CONSIDER A DOMINO ANALOGY:



INFINITELY MANY DOMINOS ARE LINED UP. $P(n)$ = 'THE n^{TH} DOMINO FALLS'. WE WISH TO PROVE $\forall n\, P(n)$ = 'ALL DOMINOS FALL'.

I. SHOW $P(1)$ = 'THE 1^{ST} DOMINO FALLS'
II. SHOW $\forall n\,(P(n) \to P(n+1))$ i.e. 'IF ANY PARTICULAR DOMINO FALLS, THEN THE NEXT DOMINO ALSO FALLS.'

WE CONCLUDE FROM I & II THAT $\forall n\, P(n)$ i.e. 'ALL DOMINOS FALL'.

THE VALIDITY OF THIS PROOF TECHNIQUE IS BASED ON THE FOLLOWING

<u>THEOREM</u>: PRINCIPLE OF MATHEMATICAL INDUCTION (PMI)
FOR ANY PROPOSITIONAL FUNCTION $P: \mathbb{Z}^+ \to \{F, T\}$ THE FOLLOWING IS A TAUTOLOGY:

$$\left[P(1) \wedge \forall n\,(P(n) \to P(n+1))\right] \to \forall n\, P(n).$$

PROOF LATER.

EX. $\forall n \geq 1 \quad \sum_{K=1}^{n} (2K-1) = n^2$

PROOF:
LET $P(n)$ BE THE FORMULA '$\sum_{K=1}^{n} (2K-1) = n^2$'.

I. $P(1)$ is JUST $1 = 1^2$, WHICH is TRUE.

II. LET $n \geq 1$. ASSUME $P(n)$ is TRUE.
i.e. FOR THIS PARTICULAR n

$$\sum_{K=1}^{n} (2K-1) = n^2 . \quad \left\{ \text{THIS IS THE INDUCTION HYPOTHESIS} \right.$$

WE MUST SHOW $P(n+1)$ is TRUE, i.e.

$$\sum_{K=1}^{n+1} (2K-1) = (n+1)^2 .$$

WE PROCEED AS FOLLOWS:

$$\sum_{K=1}^{n+1} (2K-1) = \sum_{K=1}^{n} (2K-1) + [2(n+1)-1]$$

$$= n^2 + (2n+2-1) \quad \left\{ \text{BY THE INDUCTION HYP.} \right.$$

$$= n^2 + 2n + 1$$

$$= (n+1)^2$$

$\therefore P(n+1)$ is TRUE. SINCE $n \geq 1$ WAS CHOSEN ARBITRARILY WE'VE SHOWN $\forall n \geq 1 \; P(n) \to P(n+1)$ BY UNIVERSAL GENERALIZATION. BY P.M.I. WE CONCLUDE: $\forall n \geq 1 \; \sum_{K=1}^{n} (2K-1) = n^2$. ///

EX. LET $x \in \mathbb{R}$, $x \neq 1$. SHOW

$$\forall n \geq 1 : \boxed{\sum_{k=0}^{n-1} x^k = \frac{x^n - 1}{x - 1}} .$$

PROOF: LET $P(n)$ BE THE BOXED STATEMENT.

I. $P(1)$ is $1 = \frac{x-1}{x-1}$, WHICH IS TRUE

II. LET $n \geq 1$. ASSUME $P(n)$, i.e. FOR THIS n:
$$\sum_{k=0}^{n-1} x^k = \frac{x^n - 1}{x - 1}$$

WE MUST SHOW $P(n+1)$ IS TRUE, i.e.
$$\sum_{k=0}^{(n+1)-1} x^k = \frac{x^{n+1} - 1}{x - 1} .$$

OBSERVE
$$\sum_{k=0}^{n} x^k = \left(\sum_{k=0}^{n-1} x^k\right) + x^n$$
$$= \frac{x^n - 1}{x - 1} + x^n \quad \left\{ \text{BY THE IND. HYP.} \right.$$
$$= \frac{x^{n+1} - 1}{x - 1} \quad \left\{ \text{BY SOME ALGEBRA} \right.$$

$\therefore$ $P(n+1)$ IS TRUE $\therefore$ $\forall n \geq 1 : P(n) \Rightarrow P(n+1)$

$$\therefore \ \forall n \ \sum_{k=0}^{n-1} x^k = \frac{x^n - 1}{x - 1} .$$ ///

REMARKS

- ALWAYS STATE YOUR INDUCTION HYPOTHESIS (i.e. WHAT YOU'RE ASSUMING ON THE INDUCTION STEP) EXPLICITLY.
- ALWAYS STATE AT WHICH POINT (OR POINTS) IN YOUR PROOF THE INDUCTION HYPOTHESIS IS USED.

Ex. $\forall n \geq 1 : \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

Ex. $\forall n \geq 1 : \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

Ex. $\forall n \geq 1 : \sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$

SOMETIMES $P(n)$ IS NOT TRUE FOR ALL $n \geq 1$, i.e. SOME FINITE NUMBER OF INITIAL TERMS IN THE SEQUENCE $P(1), P(2), P(3), \ldots$ ARE FALSE.

TO PROVE $\forall n \geq n_0 : P(n)$ BY INDUCTION:

I. SHOW $P(n_0)$ IS TRUE

II. SHOW $\forall n \geq n_0 : P(n) \rightarrow P(n+1)$.

EX. $\forall n \geq 4$ $\boxed{5n+8 < n^2+4n+1}$

PROOF

LET $P(n)$ BE THE BOXED INEQUALITY ABOVE.
(NOTE $P(1)$, $P(2)$, $P(3)$ ARE ALL FALSE.)

I. $P(4)$ is $5\cdot 4+8 < 4^2+4\cdot 4+1$, i.e. $28 < 33$, WHICH IS TRUE.

II. LET $n \geq 4$. ASSUME $P(n)$ IS TRUE, i.e. FOR THIS n : $5n+8 < n^2+4n+1$.
WE MUST SHOW $P(n+1)$ IS TRUE, i.e.

$$5(n+1)+8 < (n+1)^2+4(n+1)+1.$$

THUS

$$\begin{aligned} 5(n+1)+8 &= (5n+8)+5 \\ &< (n^2+4n+1)+5 \quad \text{(BY IND. HYP.)} \\ &= n^2+4n+6 \\ &\leq (n^2+4n+6)+2n \quad \text{(SINCE } n\geq 4 \Rightarrow 2n\geq 0\text{)} \\ &= n^2+6n+6 \\ &= (n+1)^2+4(n+1)+1 \quad \text{(BY SOME ALGEBRA).} \end{aligned}$$

$\therefore$ $P(n+1)$ is TRUE.

$\therefore$ $\forall n \geq 4$: $5n+8 < n^2+4n+1$ BY P.M.I.

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