THEOREM

IF S IS FINITE THEN SO IS $\mathbb{P}(S)$ AND

$$|\mathbb{P}(S)| = 2^{|S|}.$$

(HENCE THE NAME 'POWER' SET.)

EX. $S = \{1,2,3\}$

$$\mathbb{P}(S) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, S\}$$

THUS $|\mathbb{P}(S)| = 8 = 2^3 = 2^{|S|}$.

RECALL A SET IS AN UNORDERED COLLECTION. AN ORDERED COLLECTION OF n OBJECTS IS CALLED AN ORDERED n-TUPLE:

NOTATION: $(x_1, x_2, x_3, \ldots, x_n)$

IF $n=2$ WE SPEAK OF AN ORDERED PAIR.

NOTE $(x_1, x_2) = (y_1, y_2)$ IFF $x_1 = y_1$ AND $x_2 = y_2$.
FOR INSTANCE $(u,v) = (1,2)$ IFF $u=1$, $v=2$.

DEFN

THE CARTESIAN PRODUCT OF TWO SETS A, B IS THE SET OF ALL ORDERED PAIRS (x,y) WHERE $x \in A$ AND $y \in B$.
NOTATION: $A \times B$.

$$A \times B = \{(x,y) \mid x \in A \wedge y \in B\}$$

EX. $A = \{1,2,3\}$, $B = \{a,b\}$

$$A \times B = \{(1,a),(1,b),(2,a),(2,b),(3,a),(3,b)\}$$

THEOREM

IF A AND B ARE FINITE SO IS $A \times B$, AND

$$|A \times B| = |A| \cdot |B|.$$

MORE GENERALLY THE CARTESIAN PRODUCT OF n SETS $A_1, A_2, \ldots, A_n$ IS

$$A_1 \times A_2 \times \cdots \times A_n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in A_i \text{ FOR } 1 \le i \le n\}$$

AND

$$|A_1 \times A_2 \times \cdots \times A_n| = |A_1| \cdot |A_2| \cdots |A_n|.$$

RUSSEL'S PARADOX (PROB 30 P. 86)

LET $\mathcal{S} = \{\text{ALL SETS}\}$. OBSERVE $\mathcal{S} \in \mathcal{S}$.
BUT $\phi \notin \phi$, SO SOME SETS HAVE THEMSELVES AS MEMBERS, AND SOME DO NOT.

DEFINE: $\mathcal{R} = \{ A \in \mathcal{S} \mid A \notin A \}$

IS $\mathcal{R} \in \mathcal{R}$ OR $\mathcal{R} \notin \mathcal{R}$?

OBSERVE THAT $\mathcal{R} \in \mathcal{R} \rightarrow \mathcal{R} \notin \mathcal{R}$ AND $\mathcal{R} \notin \mathcal{R} \rightarrow \mathcal{R} \in \mathcal{R}$. THUS

$$\mathcal{R} \in \mathcal{R} \text{ IFF } \mathcal{R} \notin \mathcal{R}$$

WE AVOID THIS PARADOX BY SIMPLY NOT ALLOWING THE CONSTRUCTION OF $\mathcal{S}$, OR ANY OTHER SET WHICH IS A MEMBER OF ITSELF. THIS AD-HOC APPROACH IS OFTEN CALLED <u>NAIEVE SET THEORY</u>.

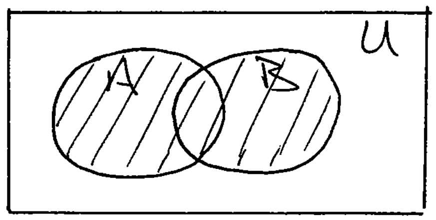
(1.7) SET OPERATIONS

DEFN:

THE UNION OF SETS A, B IS THE SET

$$A \cup B = \{x \in U \mid x \in A \vee x \in B\}$$

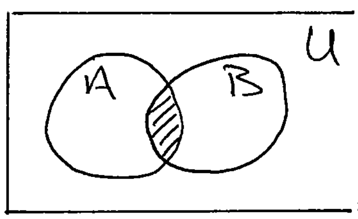
VENN DIAGRAM:



DEFN:

THE INTERSECTION OF SETS A, B IS THE SET

$$A \cap B = \{x \in U \mid x \in A \wedge x \in B\}$$

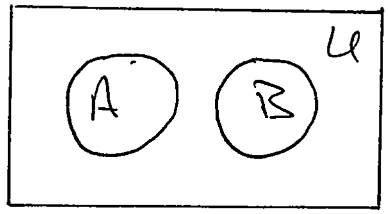


Ex. $A = \{1, 2, 3\}$, $B = \{2, 3, 4, 5\}$

$$A \cup B = \{1, 2, 3, 4, 5\}$$

$$A \cap B = \{2, 3\}$$

WE CALL A AND B DISJOINT IF $A \cap B = \emptyset$



IF A AND B ARE FINITE, THEN SO ARE $A \cup B$ AND $A \cap B$; AND

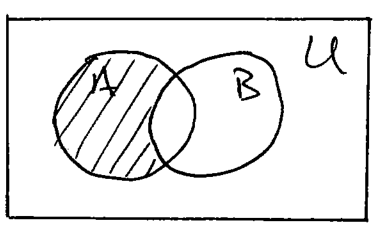
$$|A \cup B| = |A| + |B| - |A \cap B|$$

THIS IS A SPECIAL CASE OF THE <u>PRINCIPLE OF INCLUSION-EXCLUSION</u> (PIE), WHICH IS AN IMPORTANT COUNTING TECHNIQUE

<u>DEFN</u>

THE <u>SET DIFFERENCE</u> $A - B$ IS THE SET

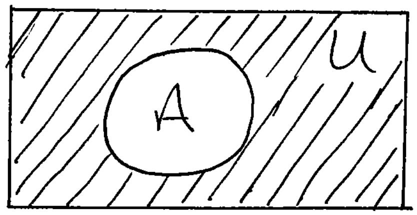
$$A - B = \{ x \in U \mid x \in A \wedge x \notin B \}$$



DEFN.

THE COMPLEMENT OF A IS THE SET

$$\overline{A} = U - A = \{x \in U \mid x \notin A\}$$



EX. $U = \{1, \ldots, 10\}$, $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$

$$A - B = \{1, 2\}, \quad \overline{A} = \{5, \ldots, 10\}$$

NOTE $\overline{A}$ IS MEANINGLESS UNLESS U HAS BEEN SPECIFIED.

READ SET IDENTITIES TABLE 1, P. 89

$$\left.\begin{aligned} A \cup (B \cup C) &= (A \cup B) \cup C \\ A \cap (B \cap C) &= (A \cap B) \cap C \end{aligned}\right\} \text{ ASSOCIATIVE LAWS}$$

$$\left.\begin{aligned} A \cap (B \cup C) &= (A \cap B) \cup (A \cap C) \\ A \cup (B \cap C) &= (A \cup B) \cap (A \cup C) \end{aligned}\right\} \text{ DISTRIBUTIVE LAWS}$$

$$\left.\begin{aligned} \overline{A \cup B} &= \overline{A} \cap \overline{B} \\ \overline{A \cap B} &= \overline{A} \cup \overline{B} \end{aligned}\right\} \text{ DEMORGAN'S LAWS}$$

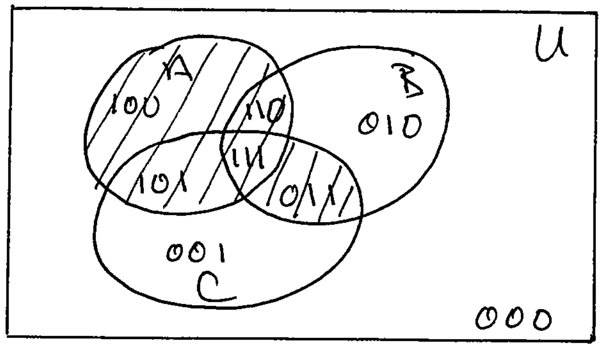
PROOF OF 1ST DEMORGAN:

$$\overline{A \cup B} = \{x \mid x \notin A \cup B\}$$
$$= \{x \mid \neg\, x \in A \cup B\}$$
$$= \{x \mid \neg\,(x \in A \vee x \in B)\}$$
$$= \{x \mid \neg\, x \in A \wedge \neg\, x \in B\}$$
$$= \{x \mid x \notin A \wedge x \notin B\}$$
$$= \{x \mid x \in \overline{A} \wedge x \in \overline{B}\}$$
$$= \overline{A} \cap \overline{B}$$

IDENTITIES CAN ALSO BE PROVED BY MEMBERSHIP TABLES, e.g. 2ND DISTRIBUTIVE

A	B	C	$B \cap C$	$A \cup (B \cap C)$	$A \cup B$	$A \cup C$	$(A \cup B) \cap (A \cup C)$
0	0	0	0	0	0	0	0
0	0	1	0	0	0	1	0
0	1	0	0	0	1	0	0
0	1	1	1	1	1	1	1
1	0	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	1	0	0	1	1	1	1
1	1	1	1	1	1	1	1

Thus $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.



NOTATION:

GIVEN SETS $A_1, A_2, \ldots, A_n$

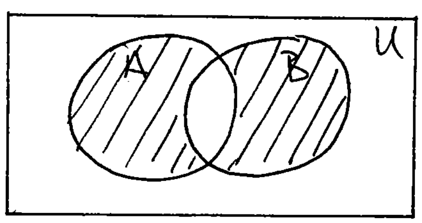
$$A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{i=1}^{n} A_i$$

$$A_1 \cap A_2 \cap \cdots \cap A_n = \bigcap_{i=1}^{n} A_i$$

DEFN

THE SYMMETRIC DIFFERENCE OF A, B IS

$$A \oplus B = \{ x \in U \mid x \in A \oplus x \in B \}$$



EX. SHOW $A \oplus B = (A \cup B) - (A \cap B)$.

(1.8) FUNCTIONS

DEFN

A FUNCTION CONSISTS OF THREE THINGS:

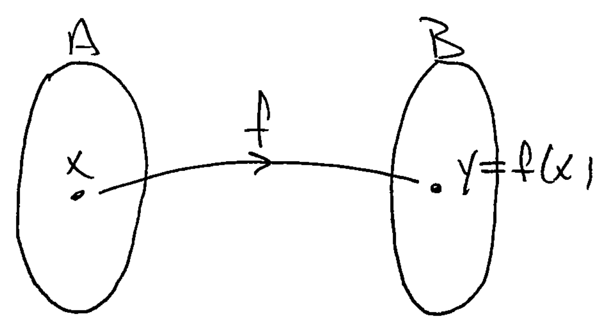
1.) A SET A CALLED THE <u>DOMAIN</u>

2.) A SET B CALLED THE <u>CODOMAIN</u>

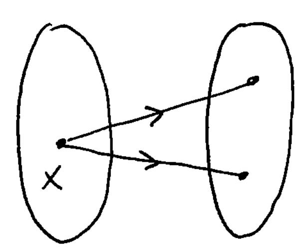
3.) A RULE f WHICH ASSIGNS TO EACH ELEMENT OF A, A <u>UNIQUE</u> ELEMENT OF B.

IF $x \in A$ IS ASSIGNED TO $y \in B$ WE WRITE $y = f(x)$. y IS CALLED THE <u>IMAGE</u> OF x UNDER f AND x IS CALLED THE <u>PREIMAGE</u> OF y UNDER f

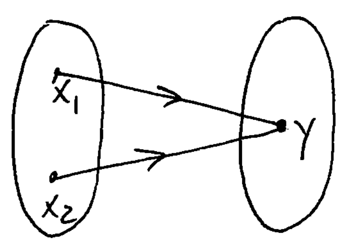
NOTATION: $f: A \to B$



UNIQUENESS:



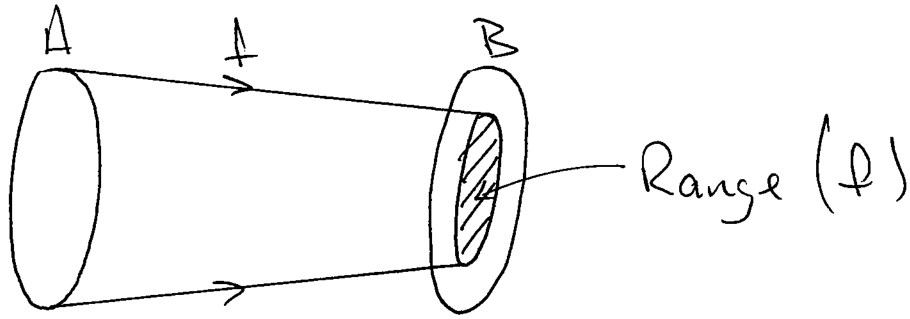
NOT A FUNCTION



OK

THE *RANGE* OF f IS THE SET OF ALL IMAGES UNDER f.

$$\text{Range}(f) = \{ y \in B \mid \exists x \in A : y = f(x) \}$$
$$= \{ f(x) \mid x \in A \}$$



Ex. $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = x^2$

$$\text{Range}(f) = \{0, 1, 4, 9, 16, 25, \ldots\}$$

Ex. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$

$$\text{Range}(f) = [0, \infty) = \{ y \in \mathbb{R} \mid y \geq 0 \}.$$

DEFN
LET $f : A \to B$, $S \subseteq A$. THE *IMAGE* OF S UNDER f IS

$$f(S) = \{ f(x) \mid x \in S \}.$$

NOTE: $f(S) \subseteq f(A) = \text{Range}(f) \subseteq B$.