WE NOW CONCENTRATE ON 2^{nd} ORDER LINEAR, HOMOGENEOUS, CONST. COEF.

(1) $$x_n = c_1 x_{n-1} + c_2 x_{n-2}$$

THEOREM (SUPERPOSITION PRINCIPLE.)
IF (a_n) AND (b_n) ARE SOLUTIONS TO (2), THEN SO IS $(\alpha a_n + \beta b_n)$ FOR ANY CONSTANTS α, β.

PROOF:
LET $x_n = \alpha a_n + \beta b_n$. THEN

$$\begin{aligned}
RHS &= c_1 x_{n-1} + c_2 x_{n-2} \\
&= c_1(\alpha a_{n-1} + \beta b_{n-1}) + c_2(\alpha a_{n-2} + \beta b_{n-2}) \\
&= \alpha(c_1 a_{n-1} + c_2 a_{n-2}) + \beta(c_1 b_{n-1} + c_2 b_{n-2}) \\
&= \alpha a_n + \beta b_n \\
&= x_n \\
&= LHS.
\end{aligned}$$

///

THIS FACT WILL BE HELPFUL IN FINDING SOLUTIONS TO (1).

TO SOLVE (1) WE GUESS A SOLUTION (BASED ON PAST EXPERIENCE) OF THE FORM

$$x_n = r^n$$

FOR SOME CONSTANT r. SUBSTITUTING INTO (1) YIELDS

$$r^n = c_1 r^{n-1} + c_2 r^{n-2}.$$

DIVIDING THROUGH BY r^{n-2} WE GET

(2) $$r^2 = c_1 r + c_2$$

OR

(3) $$r^2 - c_1 r - c_2 = 0$$

(2) OR (3) IS CALLED THE <u>CHARACTERISTIC EQUATION</u> OF (1). IF r_0 IS ANY ROOT OF (3) THEN $x_n = r_0^n$ IS A SOLUTION TO (1).

<u>THEOREM</u>

IF r_0 IS ANY ROOT OF THE CHARACTERISTIC EQUATION (2) OR (3) THEN $x_n = r_0^n$ IS A SOLUTION TO (1).

PROOF:

LET $x_n = r_0^n$. THEN SUBSTITUTING INTO (1)

$$
\begin{aligned}
RHS &= C_1 x_{n-1} + C_2 x_{n-2} \\
&= C_1 r_0^{n-1} + C_2 r_0^{n-2} \\
&= r_0^{n-2}(C_1 r_0 + C_2) \\
&= r_0^{n-2} \cdot r_0^2 \qquad \text{(SINCE } r_0 \text{ IS ROOT OF (2))} \\
&= r_0^n \\
&= x_n \\
&= LHS.
\end{aligned}
$$

///

THUS THE KEY TO FINDING SOLUTIONS TO (1) IS FINDING ROOTS OF THE CHARACTERISTIC EQUATION (3). THE ROOTS OF (3) ARE

$$r_1 = \frac{C_1 + \sqrt{C_1^2 + 4C_2}}{2}, \quad r_2 = \frac{C_1 - \sqrt{C_1^2 + 4C_2}}{2}$$

THERE ARE THREE CASES TO CONSIDER

(i) $r_1 \neq r_2$ ARE DISTINCT AND REAL $(C_1^2 + 4C_2 > 0)$.

(ii) $r_1 = r_2$ ARE REPEATED REAL ROOTS $(C_1^2 + 4C_2 = 0)$.

(iii) $r_1 \neq r_2$ ARE COMPLEX CONJUGATES $(C_1^2 + 4C_2 < 0)$.

WE CONSIDER ONLY CASES (i) AND (ii).

THEOREM (CASE (i))

IF r_1, r_2 ARE DISTINCT REAL ROOTS OF THE CHARACTERISTIC EQUATION (3), THEN EVERY SOLUTION TO (1) IS OF THE FORM

$$x_n = \alpha r_1^n + \beta r_2^n$$

FOR SOME CONSTANTS α, β (DETERMINED BY INITIAL CONDITIONS.)

Ex. $a_n = 5a_{n-1} - 6a_{n-2}$

$a_0 = 1,\ a_1 = 0$

Soln: $a_n = 3 \cdot 2^n - 2 \cdot 3^n$

Ex. $x_n = 4x_{n-2}$, $x_0 = 0,\ x_1 = 4$

Soln: $x_n = 2^n - (-2)^n = \begin{cases} 2^{n+1} & n \text{ ODD} \\ 0 & n \text{ EVEN} \end{cases}$

Ex. (FIBONACCI)

$F_n = F_{n-1} + F_{n-2}$, $F_0 = 0,\ F_1 = 1.$

Soln:

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

PROOF:
LET $(a_n)_{n=0}^{\infty}$ BE ANY SOLUTION TO (1).
DEFINE CONSTANTS α, β BY

$$\alpha = \frac{a_1 - r_2 a_0}{r_1 - r_2}, \quad \beta = \frac{r_1 a_0 - a_1}{r_1 - r_2}$$

WE'VE ALREADY SHOWN THAT $x_n = \alpha r_1^n + \beta r_2^n$ IS A SOLUTION TO (1). IT REMAINS ONLY TO CHECK THAT THE INITIAL CONDITIONS

$$x_0 = a_0 \quad \text{AND} \quad x_1 = a_1$$

ARE SATISFIED. WE LEAVE THIS AS AN EXERCISE. SINCE THE INITIAL TERMS UNIQUELY DETERMINE A SOLUTION, IT FOLLOWS THAT $x_n = a_n$ IS OF THE REQUIRED FORM.

///.

THEOREM (CASE (ii))
SUPPOSE THE CHARACTERISTIC EQUATION (3) HAS JUST ONE REAL ROOT r_0. THEN EVERY SOLUTION TO (1) IS OF THE FORM

$$x_n = \alpha r_0^n + \beta n r_0^n$$

FOR SOME CONSTANTS α, β (DETERMINED BY INITIAL CONDITIONS.)

WE LEAVE THE PROOF AS AN EXERCISE.

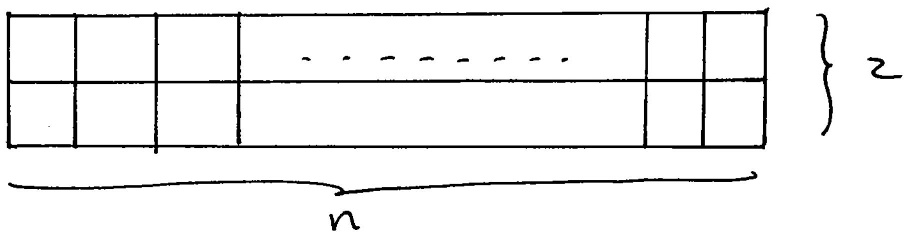
<u>Ex.</u> $b_n = -4b_{n-1} - 4b_{n-2}$, $b_0 = 0, b_1 = 1$

Soln: $b_n = (-1)^{n-1} n 2^{n-1}$

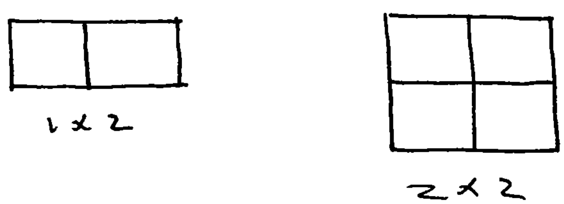
<u>Ex.</u>

IN HOW MANY WAYS CAN A $(2 \times n)$ RECTANGULAR WALK WAY BE TILED USING ONLY (1×2) AND (2×2) TILES?

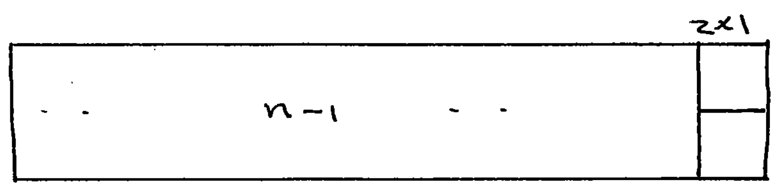
WALK:



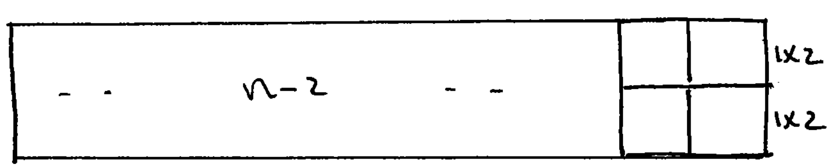
TILES:



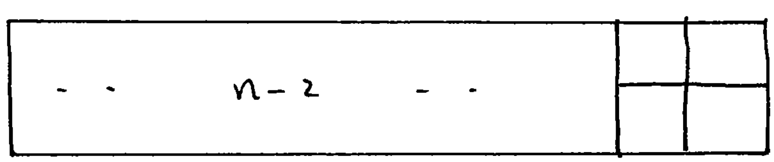
THERE ARE 3 (MUTUALLY EXCLUSIVE) WAYS THE WALK CAN END:

1.) 

a_{n-1} WAYS

2.) 

a_{n-2} WAYS

3.) 

a_{n-2} WAYS

EVERY WALK IS OF TYPE (1), (2), OR (3). IF a_n IS THE NUMBER OF SUCH TILINGS THEN

$$a_n = a_{n-1} + 2a_{n-2}$$

WITH

$$a_1 = 1, \quad a_2 = 3$$

SOLN: $a_n = \frac{1}{3}\left(2^{n+1} + (-1)^n\right)$.

<u>READ</u>

- DISTINCT ROOTS, HIGHER ORDER, THM 3, P.417
- REPEATED ROOTS, HIGHER ORDER, THM 4, P.418
- NON HOM. EQNS. P.419-422.