6.1 Recurrence Relations

Recall that a <u>k^{th} order recurrence relation</u> (or just recurrence) is an equation which gives the n^{th} term of a sequence as a function of the k preceding terms:

(1) $$x_n = f(x_{n-1}, x_{n-2}, \ldots, x_{n-k})$$

If k <u>initial terms</u> $x_0, x_1, \ldots, x_{k-1}$ are specified, then such an expression uniquely determines the sequence $(x_n)_{n=0}^{\infty}$.

A sequence $(a_n)_{n=0}^{\infty}$ which when substituted into the left and right sides of (1) results in an identity for all $n \geq 0$ is said to <u>solve</u> or <u>satisfy</u> the recurrence (1).

i.e. (a_n) is a <u>solution</u> to (1) iff

$$\forall n \geq 0 : \; a_n = f(a_{n-1}, \ldots, a_{n-k}).$$

In general a recurrence such as (1) has infinitely many solutions.

EX. $x_n = 5x_{n-1} - 6x_{n-2}$

WE SHOW THAT $a_n = 2^n$ SOLVES THIS RECURRENCE.

$$\begin{aligned} \text{RHS} &= 5a_{n-1} - 6a_{n-2} \\ &= 5\cdot 2^{n-1} - 6\cdot 2^{n-2} \\ &= (5\cdot 2 - 6)2^{n-2} \\ &= 4\cdot 2^{n-2} \\ &= 2^n \\ &= a_n \\ &= \text{LHS} \end{aligned}$$

EXERCISE.: PROVE THAT $b_n = 3^n$ ALSO SOLVES THIS RECURRENCE, AND IN FACT SO DOES ANY SEQUENCE OF THE FORM $(\alpha\cdot 2^n + \beta\cdot 3^n)$ FOR ANY CONSTANTS α AND β.

EX. THE FIBONACCI RECURRENCE $F_n = F_{n-1} + F_{n-2}$.

THE UNIQUE SEQUENCE SATISFYING THIS RECURRENCE AND THE INITIAL CONDITIONS $F_0 = 0$, $F_1 = 1$ IS:

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

EXERCISE: PROVE THE ABOVE STATEMENT.
(HINT: USE THE FACT THAT $\frac{1\pm\sqrt{5}}{2}$ ARE ROOTS OF THE QUADRATIC: $r^2 - r - 1 = 0$.)

THE QUANTITY $\frac{1+\sqrt{5}}{2}$ IS OFTEN CALLED THE <u>GOLDEN RATIO</u>.

RECURRENCES MAY BE USED TO MODEL PROBLEMS IN ECONOMICS AND POPULATION GROWTH.

<u>EX</u> \$5000 IS PLACED IN AN ACCOUNT EARNING 6% APR COMPOUNDED MONTHLY. WHAT AMOUNT WILL BE IN THE ACCOUNT AFTER 10 YEARS?

DEFINE

$$A_n = \text{AMOUNT (\$) AFTER } n \text{ MONTHS}$$

THEN

$$A_0 = 5000$$

AND

$$A_n = A_{n-1} + \left(\frac{.06}{12}\right) A_{n-1} = (1.005) A_n .$$

THUS

$$\begin{aligned} A_1 &= (1.005) \cdot 5000 \\ A_2 &= (1.005) \cdot (1.005) 5000 = (1.005)^2 5000 \\ A_3 &= (1.005)(1.005)^2 5000 = (1.005)^3 5000 \\ &\vdots \\ A_n &= (1.005)^n \cdot 5000 . \end{aligned}$$

<u>EXERCISE</u>: PROVE THAT $\left((1.005)^n \cdot 5000\right)_{n=0}^{\infty}$ SOLVES THE RECURRENCE $A_n = (1.005) \cdot A_{n-1}$ AND SATISFIES $A_0 = 5000$.

AFTER 10 YEARS (120 MONTHS) WE HAVE

$$A_{120} = (1.005)^{120} \cdot 5000 = \$\, 9096.98 .$$

RECURRENCES CAN ALSO BE USED TO MODEL COUNTING PROBLEMS.

EX. LET B_n DENOTE THE NUMBER OF BIT STRINGS OF LENGTH n WHICH CONTAIN 2 CONSECUTIVE ZEROS. FIND A RECURRENCE FOR THE SEQUENCE $(B_n)_{n=0}^{\infty}$.

FIRST OBSERVE: $B_0 = 0, B_1 = 0, B_2 = 1, B_3 = 3, \ldots$

WE CAN CONSTRUCT A BIT STRING OF LENGTH n WHICH CONTAINS 2 CONSECUTIVE ZEROS BY PERFORMING EXACTLY ONE OF THE FOLLOWING SUBTASKS.

(1) FORM A BIT STRING OF LENGTH $n-1$ WHICH CONTAINS 2 CONSECUTIVE ZEROS, THEN APPEND A 1.

$$\underbrace{x\,x \ldots x}_{n-1}1 \; : \; B_{n-1} \text{ WAYS}$$

(2) FORM A BIT STRING OF LENGTH $n-2$ WHICH CONTAINS 2 CONSECUTIVE ZEROS, THEN APPEND 10.

$$\underbrace{xx\cdots x}_{n-2}10 \;:\; B_{n-2} \text{ WAYS}$$

(3) FORM AN ARBITRARY BIT STRING OF LENGTH $n-2$, THEN APPEND 00

$$\underbrace{xx\cdots x}_{n-2}00 \;:\; 2^{n-2} \text{ WAYS}$$

NOTE THAT EVERY BIT STRING OF LENGTH n CONTAINING 00 FALLS INTO EXACTLY ONE OF THE ABOVE CLASSES. BY THE SUM RULE:

$$B_n = B_{n-1} + B_{n-2} + 2^{n-2}$$

USING INITIAL TERMS $B_0 = B_1 = 0$ WE OBTAIN.

$$\begin{aligned}
B_2 &= 0 + 0 + 2^0 = 1 \\
B_3 &= 1 + 0 + 2^1 = 3 \\
B_4 &= 3 + 1 + 2^2 = 8 \\
B_5 &= 8 + 3 + 2^3 = 19 \\
B_6 &= 19 + 8 + 2^4 = 43 \\
&\vdots
\end{aligned}$$

EXERCISE

SHOW THAT

$$B_n = \left(\frac{3-\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n - \left(\frac{3+\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n + 2^n$$

IS THE UNIQUE SOLUTION TO $B_n = B_{n-1} + B_{n-2} + 2^{n-1}$ SATISFYING $B_0 = B_1 = 0$.

(HINT: AS FOR THE FIBONACCI SEQUENCE, USE THE FACT THAT $\frac{1 \pm \sqrt{5}}{2}$ ARE ROOTS OF THE QUADRATIC $r^2 - r - 1 = 0$.)

6.2 SOLVING LINEAR RECURRENCE RELATIONS

A K^{TH} ORDER LINEAR HOMOGENEOUS RECURRENCE RELATION WITH CONSTANT COEFFICIENTS IS AN EQUATION OF THE FORM

$$x_n = c_1 x_{n-1} + c_2 x_{n-2} + \cdots + c_K x_{n-K}$$

WHERE $c_1, \ldots, c_K \in \mathbb{R}$ AND $c_K \neq 0$.

- LINEAR
  RHS IS A 1^{ST} DEGREE POLYNOMIAL IN $x_{n-1}, \ldots, x_{n-K}$
- HOMOGENEOUS
  EACH TERM ON RHS IS OF DEGREE 1
- CONSTANT COEFFICIENTS
  $c_1, \ldots, c_K$ ARE NOT FUNCTIONS OF n.

EXAMPLES

	ORDER	LINEAR	HOMOGENEOUS	CONST. COEF.
$F_n = F_{n-1} + F_{n-2}$	2	yes	yes	yes
$x_n = 2x_{n-1}$	1	yes	yes	yes
$B_n = B_{n-1} + B_{n-2} + 2^{n-2}$	2	yes	no	yes
$y_n = y_{n-1} + y_{n-2}^2$	2	no	NA	NA
$a_n = n \cdot a_{n-1} + n$	1	yes	no	no