4.5 Generalized Permutations & Combinations

Recall some formulas derived from previous examples.

The number of strings of length k from an alphabet of size n is n^k, while the number of such strings with no repeated symbols is

$$n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}$$

Both formulas follow immediately from the product rule.

The words 'string' and 'permutation' have slightly different meanings. Both refer to ordered arrangements but:

- Permutations & strings without repetition
- Permutations with repetition & strings

Thus 'permutation' means that no element is repeated, while 'string' means implicitly that repetition is allowed.

We may occasionally refer to 'permutations without repetition' or 'strings with

REPETITION' EVEN THOUGH THESE PHRASES ARE REDUNDANT.

PERMUTATIONS WITH REPETITION (i.e. STRINGS) ARE SOMETIMES CALLED <u>GENERALIZED PERMUTATIONS</u>. SUMMARIZING WE HAVE:

- THE NUMBER OF K-PERMUTATIONS FROM AN n ELEMENT SET (WITHOUT REPETITION):

$$P(n,k) = \frac{n!}{(n-k)!} .$$

- THE NUMBER OF K-PERMUTATIONS FROM AN n ELEMENT SET, WITH REPETITION:

$$n^K .$$

<u>COMBINATIONS WITH REPETITION</u>

WE MAY ALSO CONSIDER UNORDERED ARRANGEMENTS IN WHICH ELEMENTS MAY BE REPEATED. THESE ARE CALLED <u>COMBINATIONS WITH REPETITION</u> (OR <u>WITH REPLACEMENT</u>.)

<u>EX.</u> AN URN CONTAINS ONE RED BALL, ONE BLUE BALL, AND ONE GREEN BALL. WE DRAW 4 BALLS FROM THE URN, EACH TIME NOTING IT'S COLOR, THEN REPLACING IT. HOW MANY DIFFERENT COLOR COMBINATIONS ARE POSSIBLE?

NOTE: THE ORDER IN WHICH COLORS OCCUR IS NOT RELEVANT, i.e. RRBG IS THE SAME AS GRBR.

$$\left.\begin{array}{l} RRRR, BBBB, GGGG, \\ RRRB, RRRG, \\ GGGR, GGGB, \\ BBBR, BBBG, \\ RRBB, RRGG, BBGG, \\ RRBG, BBRG, GGRB \end{array}\right\} \text{15 COMBINATIONS}$$

WE HAVE COUNTED THE NUMBER OF 4-COMBINATIONS FROM A 3 ELEMENT SET $\{R, B, G\}$, WHERE REPETITION IS ALLOWED.

NOTICE IN THIS EXAMPLE, THAT EACH SUCH COMBINATION CORRESPONDS TO A STRING CONSISTING OF 4 STARS (*) AND 2 VERTICAL BARS (|).

$$\begin{array}{lcl} |*|*** & \longleftrightarrow & BGGG \\ **||** & \longleftrightarrow & RRGG \\ *|***| & \longleftrightarrow & RBBB \end{array}$$

THE 2 BARS PARTITION THE STRING INTO 3 CELLS. THE NUMBER OF STARS IN THE FIRST CELL IS THE NUMBER OF

RED BALLS DRAWN, THE SECOND CELL GIVES THE NUMBER OF BLUE BALLS, AND THE THIRD CELL GIVES THE GREEN BALLS.

MOREOVER THIS CORRESPONDENCE IS ACTUALLY A BIJECTION, EACH COLOR COMBINATION CORRESPONDS TO EXACTLY ONE STRING, AND EACH STRING TO EXACTLY ONE COLOR COMBINATION.

THUS WE MAY COUNT STRINGS INSTEAD. SUCH A STRING IS SPECIFIED WHEN WE CHOOSE WHICH 4 OF THE 6 POSITIONS ARE TO BE OCCUPIED BY STARS

$$\underset{1}{\underline{*}}\quad \underset{2}{\underline{|}}\quad \underset{3}{\underline{*}}\quad \underset{4}{\underline{|}}\quad \underset{5}{\underline{*}}\quad \underset{6}{\underline{*}}$$

(ALTERNATELY WE COULD CHOOSE WHICH 2 OF THE 6 SYMBOLS WILL BE BARS.)

THE NUMBER OF SUCH STRINGS IS

$$\binom{6}{4} = \binom{6}{2} = \frac{6!}{4!\,2!} = 15$$

WHICH IS ALSO THE NUMBER OF COLOR COMBINATIONS.

THEOREM

LET n AND k BE NON-NEGATIVE INTEGERS. THE NUMBER OF k-COMBINATIONS FROM AN n-ELEMENT SET, WHERE REPETITION IS ALLOWED, IS

$$\binom{n+k-1}{k}$$

PROOF:

LET $S = \{1, 2, \ldots, n\}$. EACH k-COMBINATION FROM S (WITH REPETITION) CORRESPONDS TO A STRING CONSISTING OF k STARS $(*)$ AND $(n-1)$ VERTICAL BARS $(|)$. THE $(n-1)$ BARS PARTITION THE STRING INTO n CELLS REPRESENTING THE ELEMENTS OF S. THE NUMBER OF STARS IN EACH CELL REPRESENTS THE NUMBER OF TIMES THE CORRESPONDING ELEMENT IS SELECTED.

MOREOVER THIS CORRESPONDENCE

$$\{k\text{-COMB. WITH REP.}\} \longleftrightarrow \{\text{STRINGS}\}$$

IS OBVIOUSLY A BIJECTION. THUS WE MAY COUNT k-COMBINATIONS FROM S WITH REPETITION BY COUNTING ALL SUCH STRINGS.

SUCH A STRING IS SPECIFIED WHEN WE CHOOSE WHICH K OF THE $n+K-1$ SYMBOLS ARE TO BE STARS.

HENCE THERE ARE $\binom{n+K-1}{K}$ SUCH STRINGS, AND THE SAME NUMBER OF K-COMBINATIONS FROM S, WITH REPETITION. ///

EX. DETERMINE THE SUM

$$\sum_{K=1}^{10} \sum_{j=1}^{K} \sum_{i=1}^{j} 1$$

EACH TERM CORRESPONDS TO A UNIQUE TRIPLE (i, j, K) SATISFYING

$$1 \le i \le j \le K \le 10$$

THE NUMBER OF SUCH TRIPLES IS THE NUMBER OF 3-COMBINATIONS FROM A 10-ELEMENT SET, WHERE REPETITION IS ALLOWED. THE SUM IS THEREFORE

$$\binom{10+3-1}{3} = \binom{12}{3} = \frac{12!}{3!\,9!} = 220 .$$

EX. A CONTRASTING EXAMPLE

$$\sum_{k=3}^{10} \sum_{j=2}^{k-1} \sum_{i=1}^{j-1} 1$$

IN THIS CASE EACH TERM CORRESPONDS TO A TRIPLE (i, j, k) SATISFYING

$$1 \leq i < j < k \leq 10 .$$

HENCE THE SUM IS THE NUMBER OF 3-COMBINATIONS FROM A 10 ELEMENT SET (WITHOUT REPETITION)

$$\binom{10}{3} = \frac{10!}{3!\,7!} = 120 .$$

EX. HOW MANY SOLUTIONS HAS THE EQUATION

$$x_1 + x_2 + x_3 = 11$$

WHERE $x_i \geq 0$ AND $x_i \in \mathbb{Z}$ $(i = 1, 2, 3)$.

EACH SOLUTION CORRESPONDS TO AN 11-COMBINATION FROM THE SET $\{1, 2, 3\}$, WITH REPETITION.
(# OF 1's) $= x_1$, (# OF 2's) $= x_2$, (# OF 3's) $= x_3$.
HENCE

$$(\text{\# OF SOLUTIONS}) = \binom{3+11-1}{11} = \binom{13}{11} = 78 .$$

GIVEN K ELEMENTS FROM AN n ELEMENT SET, WE MAY FORM FOUR TYPES OF ARRANGEMENTS

		REPETITION? NO	REPETITION? YES
ORDERED?	NO (COMBINATIONS)	$\binom{n}{k} = \frac{n!}{k!(n-k)!}$	$\binom{n+k-1}{k}$
	YES (PERMUTATIONS)	$P(n,k) = \frac{n!}{(n-k)!}$	n^k

PERMUTATIONS WITH SOME OBJECTS INDISTINGUISHABLE

EX. HOW MANY ANAGRAMS OF THE WORD 'EVERGREEN' ARE THERE? (i.e. STRINGS WHICH CONTAIN EXACTLY 4 E'S, 2 R'S, AND 1 EACH OF V, G, AND N.)

EEEE
V
RR
G
N

LET m = # OF SUCH STRINGS.

NOTE LETTERS OF THE SAME TYPE ARE INDISTINGUISHABLE.

TEMPORARILY MARK THE E'S AND R'S SO AS TO DISTINGUISH BETWEEN THEM: $\{E_1, E_2, E_3, E_4\}$ AND $\{R_1, R_2\}$.

THE NUMBER OF PERMUTATIONS OF THESE (NEW) SYMBOLS IS $9!$. SUCH A PERMUTATION CAN BE CONSTRUCTED AS FOLLOWS:

- CHOOSE AN ANAGRAM OF 'EVERGREEN': m WAYS
- CHOOSE A PERMUTATION OF $\{E_1, \ldots, E_4\}$: $4!$ WAYS
- CHOOSE A PERMUTATION OF $\{R_1, R_2\}$: $2!$ WAYS.

BY THE PRODUCT RULE $9! = m \cdot 4! \cdot 2!$

$$\therefore \quad m = \frac{9!}{4!\,2!} = 7560$$

THEOREM

THE NUMBER OF PERMUTATIONS OF n OBJECTS, WHERE WE HAVE n_i INDISTINGUISHABLE OBJECTS OF TYPE i $(i = 1, 2, \ldots, k)$, AND $n_1 + n_2 + \cdots + n_k = n$, IS

$$\frac{n!}{n_1!\, n_2! \cdots n_k!}$$