4.3 PERMUTATIONS AND COMBINATIONS

DEFN: A PERMUTATION OF A SET IS SIMPLY AN ORDERED ARRANGEMENT OF ITS ELEMENTS.

EX. $S = \{1, 2, 3\}$
PERMUTATIONS OF S:

123, 132, 213, 231, 312, 321

DEFN: A K-PERMUTATION OF A SET S IS AN ORDERED ARRANGEMENT OF K ELEMENTS OF S. $(0 \le K \le |S|)$.

EX. $S = \{1, 2, 3\}$
K-PERMUTATIONS OF S, $K = 0, 1, 2, 3$:

$K=0$:	$\emptyset$	$P(3,0) = 1$
$K=1$:	1, 2, 3	$P(3,1) = 3$
$K=2$:	12, 21, 13, 31, 23, 31	$P(3,2) = 6$
$K=3$:	123, 132, 213, 231, 312, 321	$P(3,3) = 6$

NOTATION: $P(n,k)$ = # K-PERMUTATIONS OF AN n ELEMENT SET.

THEOREM

$$P(n,k) = n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!}$$

FOR $0 \le k \le n$.

THIS FOLLOWS DIRECTLY FROM THE PRODUCT RULE. NOTE THAT A K-PERMUTATION OF S IS JUST A STRING OF LENGTH K FROM THE ALPHABET S, WHERE NO SYMBOL IS REPEATED.

ALSO NOTE IF $|S| = n$, THEN AN n-PERMUTATION OF S IS JUST A PERMUTATION OF S, AND $P(n,n) = n!$

DEFN: A K-COMBINATION OF S IS AN UNORDERED COLLECTION OF K ELEMENTS OF S, i.e. A K ELEMENT SUBSET OF S. $(0 \le K \le |S|)$.

EX. $S = \{1, 2, 3\}$

K-COMBINATIONS OF S, $K = 0, 1, 2, 3$:

$K=0$:	$\emptyset$	$C(3,0) = 1$
$K=1$:	$\{1\}, \{2\}, \{3\}$	$C(3,1) = 3$
$K=2$:	$\{1,2\}, \{1,3\}, \{2,3\}$	$C(3,2) = 3$
$K=3$:	$\{1,2,3\}$.	$C(3,3) = 1$

NOTATION: $C(n,k) = \#$ K-COMBINATIONS OF AN n ELEMENT SET.

OTHER NOTATION: $\binom{n}{k}$ READ 'n-CHOOSE-K'

ALSO CALLED BINOMIAL COEFFICIENTS.

THEOREM

$$C(n,k) = \binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

FOR $0 \le k \le n$.

PROOF:
LET $|S| = n$. A k-PERMUTATION OF S CAN BE CONSTRUCTED BY PERFORMING THE FOLLOWING SUBTASKS:

- CHOOSE A k-ELEMENT SUBSET OF S IN ANY OF $C(n,k)$ WAYS
- CHOOSE A PERMUTATION OF THESE k ELEMENTS IN ANY OF $k!$ WAYS.

BY THE PRODUCT RULE

$$P(n,k) = C(n,k) \cdot k!$$

$$\therefore \quad \frac{n!}{(n-k)!} = C(n,k) \cdot k!$$

$$\therefore \quad C(n,k) = \frac{n!}{k!\,(n-k)!}$$

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EX. HOW MANY BIT STRINGS OF LENGTH 10 CONTAIN EXACTLY 6 1'S ?

$$\underset{1}{\_}\ \underset{2}{1}\ \underset{3}{\_}\ \underset{4}{1}\ \underset{5}{1}\ \underset{6}{\_}\ \underset{7}{1}\ \underset{8}{1}\ \underset{9}{1}\ \underset{10}{\_}$$

IN HOW MANY WAYS CAN WE CHOOSE WHICH 6 OF THE 10 POSITIONS WILL BE OCCUPIED BY 1'S. IN HOW MANY WAYS CAN WE CHOOSE A 6 ELEMENT SUBSET OF THE SET $\{1,2,3,4,5,6,7,8,9,10\}$?

$$C(10,6) = \frac{10!}{6!\,4!} = 210$$

NOTE WE COULD JUST AS WELL CHOOSE WHERE TO PLACE THE 4 0'S :

$$C(10,4) = \frac{10!}{4!\,6!} = 210 .$$

THEOREM

$$\binom{n}{k} = \binom{n}{n-k}$$

PROOF !

$$\binom{n}{n-k} = \frac{n!}{(n-k)!\,(n-(n-k))!} = \frac{n!}{k!\,(n-k)!} = \binom{n}{k} .$$

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THERE ARE MANY IDENTITIES INVOLVING $P(n,k)$ AND $C(n,k)$. THEY GENERALLY HAVE TWO KINDS OF PROOFS:

- ALGEBRAIC: MANIPULATE KNOWN IDENTITIES
- COMBINATORIAL: ARGUE THAT LHS AND RHS COUNT THE VERY SAME THING.

HERE IS A COMBINATORIAL PROOF OF THE PRECEDING THEOREM.

PROOF:
LET $|S| = n$, AND DEFINE FOR $0 \le k \le n$:

$$\mathcal{S}_k = \{k \text{ ELEMENT SUBSETS OF } S\} \subseteq \mathcal{P}(S),$$

DEFINE $f : \mathcal{S}_k \to \mathcal{S}_{n-k}$ BY $f(A) = \overline{A} = S - A$ FOR $A \in \mathcal{S}_k$. OBSERVE THAT $f \circ f(A) = f(f(A)) = f(\overline{A}) = \overline{\overline{A}} = A$. THUS f IS INVERTIBLE AND $f^{-1} = f$. $\therefore$ f IS A BIJECTION.

$$\therefore \quad |\mathcal{S}_k| = |\mathcal{S}_{n-k}|$$

$$\therefore \quad \binom{n}{k} = \binom{n}{n-k}. \qquad ///$$