DISCRETE MATH CE 16

WINTER 2003

(1.1) LOGIC

A PROPOSITION is a STATEMENT WHICH is (OR CAN BE EVALUATED AS) TRUE OR FALSE.

PROPOSITIONS: 'IT IS RAINING TODAY'

$2+3=5$

$2+3=7$

NOT PROPOSITIONS: 'Hello'

$x+7=2$

$a+b=c$

WE LET T AND F STAND FOR THE TRUTH VALUES TRUE AND FALSE RESPECTIVELY.

LET $p, q, r, \ldots$ BE PROPOSITIONAL VARIABLES, i.e. LETTERS WHICH STAND FOR PROPOSITIONS

WE USE LOGICAL OPERATORS TO PRODUCE NEW PROPOSITIONS FROM EXISTING ONES.

DEFN NEGATION.

'p is FALSE', 'IT IS NOT THE CASE THAT p', 'NOT p'

NOTATION: $\neg p$

IF $p =$ 'IT IS RAINING TODAY' THEN
$\neg p =$ 'IT IS NOT RAINING TODAY'.

TRUTH TABLE:

p	$\neg p$
T	F
F	T

<u>DEFN</u>: <u>CONJUNCTION</u>
'p AND q', 'BOTH p AND q ARE TRUE'
NOTATION: $p \wedge q$

p	q	$p \wedge q$
F	F	F
F	T	F
T	F	F
T	T	T

LET $p =$ 'IT IS RAINING TODAY' AND
$q =$ 'TODAY IS WEDNESDAY'. THEN
$p \wedge q =$ 'IT IS BOTH RAINING TODAY AND TODAY IS WEDNESDAY'

<u>DEFN</u>. <u>DISJUNCTION</u> (INCLUSIVE OR)
'p OR q', 'EITHER p OR q, OR POSSIBLY BOTH ARE TRUE'
NOTATION: $p \vee q$

p	q	$p \vee q$
F	F	F
F	T	T
T	F	T
T	T	T

DEFN **EXCLUSIVE OR**

'EITHER p OR q, BUT NOT BOTH ARE TRUE'

NOTATION: $p \oplus q$

p	q	$p \oplus q$
F	F	F
F	T	T
T	F	T
T	T	F

EX. 'YOUR MONEY OR YOUR LIFE'. WHICH OR IS BEING USED.

DEFN. **IMPLICATION**

'p implies q'
'IF p THEN q'
'p only if q'
'p is SUFFICIENT FOR q'
'q IF p'
'q WHENEVER p'
'q is NECESSARY FOR p'

NOTATION: $p \rightarrow q$

$p \to q$ is false only in the case when p is true and q is false

p	q	$p \to q$
F	F	T
F	T	T
T	F	F
T	T	T

<u>Ex</u> Let p = 'It is sunny today', and q = 'I will take you to the beach'.

Related Propositions:
$q \to p$ is the <u>converse</u> of $p \to q$
$\neg q \to \neg p$ is the <u>contrapositive</u> of $p \to q$

<u>Defn</u>: <u>Biconditional</u>
'p if and only if q', 'p is necessary and sufficient for q'
Notation: $p \leftrightarrow q$

$p \leftrightarrow q$ is true when p and q have the same truth value, and false otherwise

p	q	$p \leftrightarrow q$
F	F	T
F	T	F
T	F	F
T	T	T

A **COMPOUND PROPOSITION** IS ONE THAT IS BUILT UP FROM PROPOSITIONAL VARIABLES AND LOGICAL OPERATIONS.

Ex.

$$(p \leftrightarrow q) \vee (\neg q \rightarrow r)$$

$$\neg (p \vee q)$$

$$p \rightarrow (q \rightarrow r)$$

$$(p \rightarrow q) \rightarrow r$$

$$p \vee (q \wedge r)$$

$$(p \vee q) \wedge r$$

(1.2) LOGICAL EQUIVALENCE

A TAUTOLOGY IS A COMPOUND PROPOSITION WHICH IS TRUE FOR ANY ASSIGNMENT OF TRUTH VALUES TO ITS PROPOSITIONAL VARIABLES.

EX. $p \vee \neg p$

A CONTRADICTION IS A COMPOUND PROPOSITION WHICH IS FALSE FOR ANY ASSIGNMENT OF TRUTH VALUES TO ITS PROPOSITIONAL VARIABLES.

EX. $p \wedge \neg p$

A PROPOSITION WHICH IS NEITHER A TAUTOLOGY NOR A CONTRADICTION IS CALLED A CONTINGENCY.

DEFN

TWO COMPOUND PROPOSITIONS X, Y ARE SAID TO BE LOGICALLY EQUIVALENT IF THE PROPOSITION

$$X \Leftrightarrow Y$$

IS A TAUTOLOGY.

IN THIS CASE WE WRITE $X \equiv Y$
(OR SOMETIMES $X \Leftrightarrow Y$.)

<u>EX.</u> $p \to q \equiv \neg p \vee q$

<u>EX.</u> $p \to q \equiv \neg q \to \neg p$ (CONTRAPOSITIVE)

<u>EX.</u> $p \to q \not\equiv q \to p$ (CONVERSE).

<u>EXERCISE</u>
READ AND PROVE ALL EQUIVALENCES
ON TABLES 5, 6, 7 ON P.24

DE MORGAN'S LAWS
a.) $\neg(p \vee q) \equiv \neg p \wedge \neg q$
b.) $\neg(p \wedge q) \equiv \neg p \vee \neg q$

DISTRIBUTIVE LAWS
a.) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
b.) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

ASSOCIATIVE LAWS
a.) $p \vee (q \vee r) \equiv (p \vee q) \vee r$
b.) $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$

USING THE IDENTITIES ON TABLES 5, 6 AND 7 WE CAN NOW PERFORM 'ALGEBRAIC' MANIPULATIONS OF LOGICAL EXPRESSIONS.

Ex. SHOW $(p \vee q) \to r \equiv (p \to r) \wedge (q \to r)$

PROOF: $(p \vee q) \to r \equiv \neg(p \vee q) \vee r$ TABLE 6

$\equiv (\neg p \wedge \neg q) \vee r$ DEMORGAN

$\equiv (\neg p \vee r) \wedge (\neg q \vee r)$ DISTRIBUTIVE

$\equiv (p \to r) \wedge (q \to r)$ TABLE 6

///.

Ex. SHOW $(p \wedge q) \to (p \to q)$ is A TAUTOLOGY.

PROOF:

$(p \wedge q) \to (p \to q) \equiv \neg(p \wedge q) \vee (\neg p \vee q)$ TABLE 6

$\equiv (\neg p \vee \neg q) \vee (\neg p \vee q)$ DEMORGAN

$\equiv (\neg p \vee \neg p) \vee (\neg q \vee q)$ COMM. & ASSOC.

$\equiv \neg p \vee T$ IDEM & NEGATION

$\equiv T$ DOMINATION.

(1.3) QUATNTIFIERS

A PROPOSITIONAL FUNCTION (OR PREDICATE) $P(x)$ IS A STATEMENT WHICH BECOMES A PROPOSITION WHEN A VALUE OF x IS SPECIFIED.

EX. $P(x) =$ '$x < 7$'. WHAT IS $P(2)$, $P(10)$?

EX. $P(x) =$ 'x IS A SUNNY DAY'.

A PROPOSITIONAL FUNCTION CAN HAVE ANY NUMBER OF VARIABLES.

EX. $Q(x,y) =$ '$x + y = 10$'

THE UNIVERSE OF DISCOURSE OF A PROPOSITIONAL FUNCTION $P(x)$ IS THE SET OF POSSIBLE VALUES FOR THE VARIABLE x. WILL USUALLY WRITE U.

ONE WAY TO OBTAIN A PROPOSITION FROM A PROPOSITIONAL FUNCTION IS TO SUBSTITUTE VALUES FROM U FOR x. ANOTHER WAY IS CALLED QUANTIFICATION.

DEFN

THE <u>UNIVERSAL QUANTIFICATION</u> OF $P(x)$ IS THE STATEMENT:

'$P(x)$ IS TRUE FOR ALL x IN U'

<u>NOTATION</u>: $\forall x\, P(x)$

<u>EX</u>. $P(x) =$ '$x^2 \geq 0$' (UNIVERSE $\mathbb{R}$)

$\forall x\, P(x) =$ 'THE SQUARE OF ANY REAL NUMBER IS NON-NEGATIVE'

<u>EX</u>. $Q(x) =$ '$x < 50$' (UNIVERSE $\mathbb{R}$)

$\forall x\, Q(x) =$ 'EVERY REAL NUMBER IS LESS THAN 50'

DEFN

THE EXISTENTIAL QUANTIFICATION OF $P(x)$ IS THE STATEMENT:

'$P(x)$ IS TRUE FOR AT LEAST ONE x IN U'

<u>NOTATION</u>: $\exists x\, P(x)$