# AMS 206 – Homework Assignment #5
due Wednesday, February 14

Required Problems

1. Suppose you are observing the speeds of cars on Highway 1, and you model them as *iid* normal with unknown mean $\mu$ and unknown precision $\tau$. (When there is not traffic, this is not an unreasonable model.) Suppose you use the standard noninformative prior, $f(\mu, \tau) \propto \frac{1}{\tau}$ (this is like choosing $k = \alpha = \beta = 0$ in the parameterization from class). Suppose you observe a random sample of 40 cars and find a sample mean of $\bar{y} = 62$mph and a sample variance of $s^2 = 72$.

   (a) Find the joint posterior distribution of $\mu$ and $\tau$ by finding the posterior marginal for $\tau|\mathbf{y}$ and the posterior conditional distribution for $\mu|\tau, \mathbf{y}$. You can use the results from class, rather than re-deriving any of the equations.

   (b) Find the posterior probability that a new observation will be larger than 65. (Recall that the posterior predictive distribution is a location-scale $t$, so if you take 65, subtract the location parameter, and divide by the scale parameter (the square root of the variance parameter), you can then use R to find the probability that a standard $t$ with the appropriate degrees of freedom is larger than that standarized value.)

2. Recall that the density for the Pareto distribution is

$$f(x|\alpha, \beta) = \alpha\beta^\alpha x^{-(\alpha+1)} I_{\{x \geq \beta\}}$$

   (a) Show that the inverse CDF for the Pareto is $F^{-1}(p) = \beta(1-p)^{-1/\alpha}$.

   (b) Find the mean and variance of a Pareto distribution with parameters $\alpha = 4$ and $\beta = 5$ by Monte Carlo estimation (generate samples using the inverse CDF method), and compare them to the theoretical values of $E[X] = \frac{\alpha\beta}{\alpha-1}$ and $Var(X) = \frac{\alpha\beta^2}{(\alpha-1)^2(\alpha-2)}$.

   (c) Define a new distribution by $Y = \log(X - \beta)$ where $X$ has a Pareto($\alpha$,$\beta$) distribution (and log is the natural log). Estimate the mean and variance of this distribution for $\alpha = 4$ and $\beta = 5$ (transform your sample from the previous part).

Optional Problem

3. Show that for a normal likelihood with unknown mean and unknown precision (or variance), the marginal posterior distribution for the mean is a scaled $t$. You will probably want to use the form for the joint posterior that has the square already completed for $\mu$. Note that the density function for a scaled $t_\nu(\mu, \sigma^2)$ is

$$f(x|\nu, \mu, \sigma^2) = \frac{\Gamma\left[\frac{1}{2}(\nu+1)\right]}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\nu\pi\sigma^2}}\left[1 + \frac{1}{\nu\sigma^2}(x-\mu)^2\right]^{-(\nu+1)/2}$$