

AMS 206 – Homework Assignment #7  
due Friday, March 10

Required Problems

1. Let's revisit the cookies example with noninformative priors.
  - (a) Try using a flat prior, i.e.,  $f(\lambda) \propto I_{\{\lambda>0\}}$ , and find the posterior distribution (in closed form). What limiting Gamma prior would this be equivalent to?
  - (b) Find the posterior using Jeffreys' prior:
    - i. First, note that the likelihood for a single observation is  $f(y|\lambda) = \frac{1}{y!} \lambda^y e^{-\lambda}$ . Take the natural log of this to find the log-likelihood. Now take the second derivative with respect to  $\lambda$ .
    - ii. Find the expected value of the second derivative of the log-likelihood (in terms of  $\lambda$ ),  $E \left[ \frac{\partial^2}{\partial \lambda^2} \log f(y|\lambda) \right]$ , by using the fact that  $Y \sim Pois(\lambda)$ .
    - iii. Find the Jeffreys prior, which is proportional to the square root of the negative of the expected value of the second derivative of the log-likelihood, i.e.,
$$f(\lambda) \propto \sqrt{-E \left[ \frac{\partial^2}{\partial \lambda^2} \log f(y|\lambda) \right]}.$$
    - iv. What limiting Gamma prior would this be equivalent to?
    - v. Find the posterior distribution (in closed form) using this prior.
  - (c) For each of the noninformative priors above, as well as for the informative  $\Gamma(9, 1)$  prior used before, find the posterior mean estimate of  $\lambda$ , and compare each of these to the maximum likelihood estimate on the cookies data. How sensitive is the estimate to the choice of prior?
2. Consider again the problem of counting deer on campus, i.e., a binomial likelihood with both  $\nu$  and  $\theta$  unknown, and only a single observation  $Y$  is available. Show that bad things happen if you try to put a uniform prior on each of  $\nu$  and  $\theta$ , i.e.,  $f(\nu) \propto 1$  for all counting numbers (positive whole numbers), and  $f(\theta) = I_{\{0 \leq \theta \leq 1\}}$ . In particular:
  - (a) Find the joint posterior  $f(\nu, \theta | \mathbf{y})$ . Next find the marginal posterior for  $\nu$  by integrating out  $\theta$ , i.e.,  $f(\nu | \mathbf{y}) \propto \int_0^1 f(\nu, \theta | \mathbf{y}) d\theta$ . (You should be able to recognize that integral as a familiar pdf.)
  - (b) What happens when you try to evaluate the sum  $\sum_{\nu=0}^{\infty} f(\nu | \mathbf{y})$ ? With the right normalizing constant do you get 1?
  - (c) Using noninformative priors often gives similar results to frequentist methods. Do you get a "similar" result in this case?

Optional Problem

3. Suppose you hadn't done the math in problem 2 above, and you just blindly tried to fit that model using MCMC. Use the datum  $Y = 18$  and fit the model with MCMC. Note  $\theta$  can be sampled with a Gibbs step, while  $\nu$  requires a Metropolis-Hastings step (make sure to propose only whole numbers – one easy way to do this is to propose draws from a normal or uniform then use the `round` function). Use enough iterations, say at least 10,000. What happens? Is this consistent with your answer to #2?