ENG-27: Homework 3

1. Find the value of k for which the system,

$$\begin{aligned} x + 5y + 3z &= 0 \\ 5x + y - kz &= 0\,, \\ x + 2y + kz &= 0 \end{aligned}$$

has a nontrivial solution. For this special value of k, find the solution.

2. Using an elimination scheme, solve the equations

$$(i) \quad \begin{aligned} x_1 + x_2 &= 3 \\ 2x_1 + x_2 + x_3 &= 7 \\ x_1 + 2x_2 + 3x_3 &= 14 \end{aligned} \qquad\qquad (ii) \quad \begin{aligned} -x_1 + 2x_2 + x_3 &= 2 \\ x_2 - 2x_3 &= -3 \\ x_1 + 4x_2 - x_3 &= 4 \end{aligned}$$

3. For what values of k does the system,

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ x_1 + 2x_2 + 4x_3 &= k \ \ , \\ x_1 + 4x_2 + 10x_3 &= k^2 \end{aligned}$$

have a solution? What are the solutions for these special values of k?

4. Using the Gauss-Jordan method (i.e. row operations on an augmented matrix), invert the matrices,

$$(i) \quad A = \begin{pmatrix} -1 & 2 & 1 \\ 0 & 1 & -2 \\ 1 & 4 & -1 \end{pmatrix}, \qquad (ii) \quad A = \begin{pmatrix} 5 & 2 & 4 \\ 3 & -1 & 2 \\ 1 & 4 & -3 \end{pmatrix},$$

5. The location of a particle in a fluid can be represented by the vector $(x_1, x_2, x_3)^T$. *N.B. Transpose here simply means that the row vector becomes a column vector.* Under a certain fluid motion, it is discovered that the particle $x = (x_1, x_2, x_3)^T$ moves to the new position Ax, where

$$A = \frac{1}{9}\begin{pmatrix} 8 & -1 & -4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{pmatrix}.$$

(a) Where does the particle at $x = (2, 1, 1)^T$ move to? (b) If the new position is $(2, 1, 1)^T$, where did the particle originate? (c) Find the positions of particles that do not move in the fluid motion.

6. Find the characteristic polynomial, eigenvalues and eigenvectors of the matrices:

$$(i) \quad A = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}, \qquad (ii) \quad A = \begin{pmatrix} 1 & 1 & 3 \\ 0 & 2 & 1 \\ 4 & -4 & -1 \end{pmatrix},$$

7. Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 3 & 3 & 2 \\ 2 & 4 & 2 \\ -1 & -3 & 0 \end{pmatrix}.$$

Take the eigenvectors and create a new matrix,

$$P = \begin{pmatrix} x_1^a & x_1^b & x_1^c \\ x_2^a & x_2^b & x_2^c \\ x_3^a & x_3^b & x_3^c \end{pmatrix},$$

where $x^a = (x_1^a, x_2^a, x_3^a)^T$, $x^b = (x_1^b, x_2^b, x_3^b)^T$ and $x^c = (x_1^c, x_2^c, x_3^c)^T$ are the three eigenvectors (ignore any arbitrary constants multiplying the eigenvectors). Construct P^{-1}, and then $P^{-1}AP$. What is special about this last matrix? (This procedure is called the "diagonalization" of a matrix.)

ENG-27: Homework 3 solutions

1. $k = 1$ gives the nontrivial solution, $y = -2x$ and $z = 3x$.

2. (i) $x_1 = 1$, $x_2 = 2$ and $x_3 = 3$. (ii) $x_1 = 2$, $x_2 = 1$ and $x_3 = 2$.

3. The system has a solution if $k = 1$ or $k = 2$. For $k = 1$, the non-unique solution is $x = 2z + 1$ and $y = -3z$. For $k = 2$, the non-unique solution is $x = 2z$ and $y = 1 - 3z$.

4.

$$(i) \quad A^{-1} = \frac{1}{12}\begin{pmatrix} -7 & -6 & 5 \\ 2 & 0 & 2 \\ 1 & -6 & 1 \end{pmatrix}, \qquad (ii) \quad A^{-1} = \frac{1}{49}\begin{pmatrix} -5 & 22 & 8 \\ 11 & -19 & 2 \\ 13 & -18 & -11 \end{pmatrix},$$

5. (a) New position is $(11/9, 19/9, -2/9)^T$. (b) Old position is $(7/3, -2/3, 1/3)^T$. (c) Non-moving points: $(3t, t, -t)^T$, with t arbitrary.

6.

$$(i) \quad \lambda^2 - \lambda - 2, \qquad \lambda = 2, \quad A\begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad \lambda = -1, \quad B\begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

$$(ii) \quad (2-\lambda)(3-\lambda)(-3-\lambda) \qquad \lambda = 2, \quad A\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad \lambda = 3, \quad B\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \qquad \lambda = -3, \quad C\begin{pmatrix} 7 \\ 2 \\ -10 \end{pmatrix},$$

where A, B and C are arbitrary.

7. Eigenvalues and eigenvectors:

$$\lambda = 4, \quad A\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \qquad \lambda = 1, \quad B\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \qquad \lambda = 2, \quad C\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix},$$

The matrix P and its inverse:

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & -2 \end{pmatrix} \qquad P^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix}.$$

Then,

$$P^{-1}AP = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$