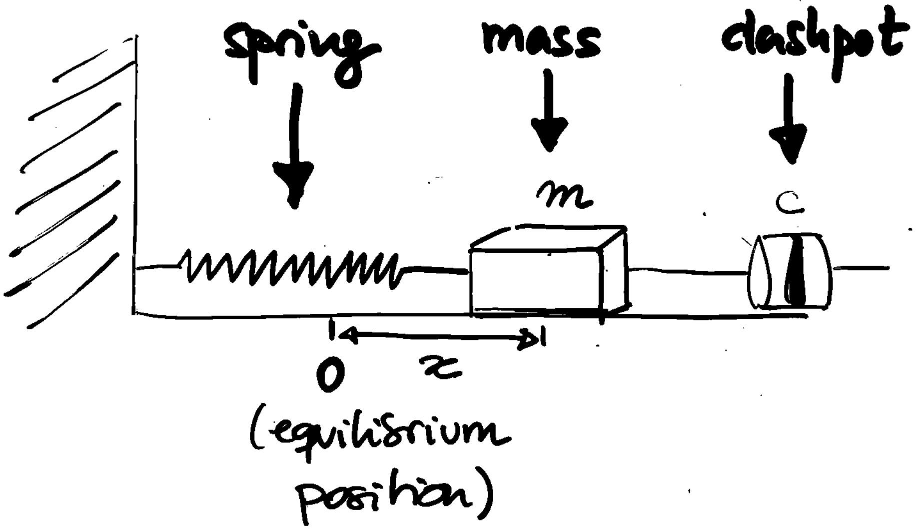
MECHANICAL VIBRATIONS



Force exerted by spring is $F_S(x) = -Kx$ ($K>0$ is the spring constant)

Force exerted by dashpot is $F_D(x) = -c\dfrac{dx}{dt}$ ($c>0$ is the damping constant)

Assume also that mass is subject to an external force $F_E = F(t)$.

Using Newton's law $F = ma$,

$$F(t) - Kx - c\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$

or

$$mx'' + cx' + Kx = F(t)$$

SECOND-ORDER LINEAR DIFF EQUATION !

Free undamped motion

$c = 0$ in $mx'' + cx' + kx = F(t)$

$F(t) = 0$

That is, $mx'' + kx = 0$.

Define $\omega_0 = \sqrt{\frac{k}{m}}$. Then we can write

$$\boxed{x'' + \omega_0^2 x = 0}$$

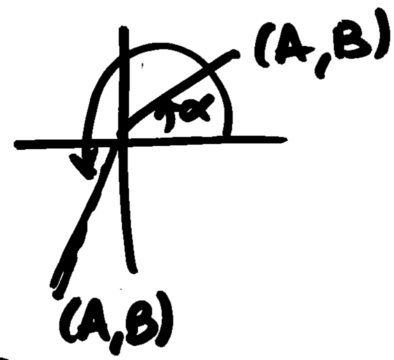
The general solution of this equation is of the form

$$x(t) = A\cos\omega_0 t + B\sin\omega_0 t$$

* Let's do a nice trick!

$$C = \sqrt{A^2 + B^2}$$

$$\alpha = \arctan(A, B)$$



Then

$$\cos\alpha = \frac{A}{C} \quad \text{and} \quad \sin\alpha = \frac{B}{C}$$

We rewrite the solution $x(t)$ as

$$x(t) = C\cdot\left(\frac{A}{C}\cos\omega_0 t + \frac{B}{C}\sin\omega_0 t\right) =$$

$$= C\cdot\left(\cos\alpha\cos\omega_0 t + \sin\alpha\sin\omega_0 t\right) =$$

$$\boxed{x(t) = C\cdot\cos(\omega_0 t - \alpha)}$$

Free damped motion

$F(t)=0$ in $mx''+cx'+kx=F(t)$

That is, $mx''+cx'+kx=0$

Define $\omega_0=\sqrt{\frac{k}{m}}$ and $p=\frac{c}{2m}>0$. Then

$$\boxed{x''+2px'+\omega_0^2x=0}$$

The characteristic equation has roots

$$r^2+2pr+\omega_0^2=0$$

$$r=-p\pm\sqrt{(p^2-\omega_0^2)}$$

- If $p^2-\omega_0^2>0$ (i.e., $c^2>4km$), then we have two distinct real roots r_1, r_2, with $r_1, r_2<0$

$$\boxed{x(t)=c_1e^{r_1t}+c_2e^{r_2t}}$$

Therefore $x(t)\xrightarrow[t\to\infty]{}0$

- If $p^2-\omega_0^2=0$ (i.e., $c^2=4km$), then we have a repeated real root $r=-p$, so

$$\boxed{x(t)=e^{-pt}(c_1)+te^{-pt}c_2}$$

Again $x(t)\xrightarrow[t\to\infty]{}0$

* If $p^2-w_0^2<0$ (i.e., $c^2<4km$), then we have two distinct complex eigenvalues $-p \pm i\sqrt{w_0^2-p^2}$. Then

$$\boxed{x(t)=e^{-pt}\left(A\cos w_1 t+B\sin w_1 t\right)}$$

where $w_1=\sqrt{w_0^2-p^2}$.

Using same trick as before, we can rewrite this as

$$\boxed{x(t)=Ce^{-pt}\cos(w_1 t-\alpha)}$$

Again $x(t)\xrightarrow[t\to\infty]{} 0$

Undamped forced motion

$$mx'' + kx = F_0 \cos\omega t$$ EXTERNAL FREQUENCY

Solutions of associated homogeneous equation are

$$x = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t \quad , \quad \omega_0 = \sqrt{\frac{k}{m}}$$

NATURAL FREQUENCY

Assume $\omega \neq \omega_0$

We try $x_p = A \cdot \cos\omega t$ and substituting, we get

$$A = \frac{F_0}{k - m\omega^2} = \frac{F_0/m}{\omega_0^2 - \omega^2}$$

So general solution is

$$x(t) = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t + \frac{F_0/m}{\omega_0^2 - \omega^2} \cos\omega t$$

RESONANCE

Note that when $\omega \to \omega_0$, $A \longrightarrow \infty$

Let's solve the equation when $\omega = \omega_0$.

[**Duplication**: $\cos\omega t$ is a solution of the associated homogeneous equation].

We try $x_p = t(A\cos\omega_0 t + B\sin\omega_0 t)$, yielding

$$x_p(t) = \frac{F_0}{2m\omega_0} t \sin\omega_0 t$$

GROWS UNBOUNDED!