The Price of Anarchy in a Network Pricing Game

John Musacchio and Shuang Wu

***Abstract*— We analyze a game theoretic model of competing network service providers that strategically price their service in the presence of elastic user demand. Demand is elastic in that it diminishes both with higher prices and congestion. The model we study is based on a model first proposed and studied by Acemoglu and Ozdaglar and later extended by Hayrapetyan, Tardos, and Wexler to consider elastic user demand. We consider the price of anarchy, which we define as the ratio of the social welfare of the system when a social planner chooses link prices versus the social welfare attained when link owners choose the link prices selfishly. Ozdaglar has recently shown that the price of anarchy in the network pricing game with elastic demand is no more than 1.5. We have independently derived the same result. In contrast to Ozdaglar's proof based on mathematical programming techniques, our proof uses linear algebra and is motivated by making an analogy to a network of resistors. Our technique is useful because it provides an intuitive explanation for the result, as well as providing a framework from which to derive extensions to the result.**

I. Motivation & Introduction

We study a a network pricing model first proposed and studied by Acemoglu and Ozdaglar [1] and later extended by Hayrapetyan, Tardos, and Wexler [2]. The model studies the pricing behavior of several providers competing to offer users connectivity between two nodes, and it has the features that a provider's link becomes less attractive as it becomes congested and that user demand is elastic – users will choose not to use any link if the sum of the price and latency of the available links is too high. In the first version of the model studied by Acemoglu and Ozdaglar, the user elasticity is modeled by assuming that all users have a single reservation utility and that if the best available price plus latency exceeds this level, users do not use any service. In this setting, the authors find that the price of anarchy – the worst case ratio of social welfare achieved by a social planner choosing prices to the social welfare arising when providers strategically choose their prices – is $(\sqrt{2}+1)/2$ [1]. (Or expressed the other way, as the ratio of welfare in Nash equilibrium to social optimum, the ratio is $2\sqrt{2}-2$.) Hayrapetyan, Tardos, and Wexler consider the model where user demand is elastic, which is the form of the model we study in the present paper [2]. They derived the first bound on the price of anarchy of this form of the model, and find a bound of 5.064. Recently, Ozdaglar has proved that the bound is actually 1.5, and furthermore that this bound is tight [3]. Ozdaglar's derivation uses techniques of mathematical programming, and is similar to the techniques used in [1]. In the present paper, we provide an independent derivation of the same result. In contrast to the proof of [3], our proof is motivated by making an analogy between the network pricing game and an electrical circuit where each branch represents a provider link and the current represents flow, and uses techniques from linear algebra. Our technique is useful because it provides an intuitive explanation for the result, as well as providing a framework from which to derive extensions to the result.

The models of [1] and [2] are part of a stream of recent literature studying the price of anarchy for games of selfish routing. Several works, see for example [4], [5], [6], study games where users are selfish, but the network is passive, the owners of the edges are not strategically choosing their prices. Other work such as [7] consider the problem of having the network choosing prices to induce optimal routing among selfish users, rather than having parts of the network selfishly choose prices to maximize revenue.

Another related work is by Johari and Tsitsiklis [8]. In [8] the authors study games where the users request a bit rate from the network, and in turn the network returns to users a price that depends on the sum of the requested rates on each resource. The model of [8] and the model of this paper study very different situations. In particular in [8] the main strategic agents are the users and in the model we study the strategic agents are link operators seeking to maximize profit. However, there are many similarities between the two games in terms of the structure of the payoff functions of the players. As in this paper, the authors of [8] derive bounds on the efficiency loss.

A. Model

We consider a network consisting of a single source and destination node. The nodes are connected by n links, each of which is owned by a distinct selfish provider. Provider i charges a price p_i per unit flow. Each link has a load dependent latency of $l_i(f_i)$ where f_i is the flow on link i and where $l_i(\cdot)$ is a convex function. We will pay particular attention to the linear case where $l_i(f_i) = a_i f_i + b_i$. The "disutility" of each link is the sum of latency and price and therefore is $l_i(f_i) + p_i$. Users are nonatomic and are free to choose the link that has the lowest disutility. Therefore, in equilibrium, all used links have a common disutility value. Users have a limit to how much disutility they will tolerate – if all the links have a disutility higher than a user's tolerance or "willingness to suffer", that user does not use any link. Users are distributed in their willingness to suffer, so we may define a function $U(x)$ to be the disutility that would induce a total flow of x across all links. To describe it another

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J. Musacchio and S. Wu are with the Technology and Information Management Program, University of California, Santa Cruz, Santa Cruz, CA 95064, USA {johnm,swu}@soe.ucsc.com

way, suppose each user has a willingness to suffer that is independent and identically distributed like the random variable W. Let S be the total population of users and let $R(d) = SP(W > d)$ be S times the complementary cumulative distribution function of willingness to suffer W. Then $U(x)$ is the inverse function of $R(d)$. Clearly $U(x)$ is decreasing. We make the additional assumption that $U(x)$ is concave. The assumption that $U(x)$ is concave is a strong assumption, but is necessary to derive the bound in this work. The authors of [3] and [2] make this assumption as well.

We refer to an instance of the network pricing game as $\mathcal{G}$. An instance $\mathcal{G}$ is specified by the collection of latency functions $\{l_i(\cdot)\}$ and the disutility function $U(\cdot)$.

We are interested in studying two configurations of the system $\mathcal{G}$. In the first configuration, the link owners adjust their prices non-cooperatively until Nash equilibrium is achieved. In the "social optimum configuration", a social planner chooses all prices to maximize social welfare. In a sense the social optimum configuration is a type of Nash equilibrium, in particular a Wardrop equilibrium, where the non atomic users are free to be strategic while the links are assigned prices by a social planner. However to avoid confusion, we will refer to this situation as simply the social optimum configuration. The social welfare of the system is defined to be the profit of providers plus the utility gathered by users. The utility gathered by users is found by integrating the difference between each unit of flow's willingness to tolerate disutility, and the disutility the flow actually bears. Therefore the utility gathered by users is

$$\int_0^f (U(x) - d)dx$$

where f is the total flow carried by the system, and d is the equilibrium disutility found on all used links.

II. Affine latency functions

Throughout this section we will assume $l_i(f_i) = a_i f_i + b_i$ and that $a_i > 0$ for all i. In the full paper we will address the case where $a_i = 0$ by using a continuity argument.

A. Price-Flow Relationships

In this subsection we find relations between the price of each link and the flow on each link in both Nash equilibrium and the social optimum configuration. By using these relations we are able to draw an analogy between the pricing game and a network of resistors. In the analogous resistor network, providers choose the resistance on their branch of the circuit. In Nash equilibrium, providers pick a resistance value that is higher than the social optimum value, causing current (flow) to be less than the socially optimal value.

We start by repeating a result found in [2], that a Nash equilibrium exists.

Lemma 1: The network pricing game with affine latency functions has a pure strategy Nash equilibrium.

Proof: The proof is found in [2]. The proof shows that players' best response functions are well defined, unique and continuous and that therefore Brouwer's fixed point theorem guarantees the existence of a Nash equilibrium. ■

Next we characterize the Nash equilibrium, the result is similar to Lemma 2.1 of [2].

Lemma 2: Consider a game $\mathcal{G}$ with linear latency functions and where the disutility function $U(\cdot)$ is continuous, concave, and everywhere differentiable. Also suppose that all links are used in the Nash equilibrium of $\mathcal{G}$. Equivalently $f_i > 0\,\forall i$. Then

$$\frac{p_i}{f_i} = a_i + \left(\sum_{j\neq i} \frac{1}{a_j} + \frac{1}{s}\right)^{-1} \tag{1}$$

where $s = -U'(f)$, and f is the total flow in Nash equilibrium.

Proof: Suppose the total flow at Nash equilibrium is f. We define $s = -U'(f)$. We now derive conditions so that no player (link owner) wants to deviate from her Nash equilibrium strategy. Suppose player i unilaterally lowers his price so that the new disutility is $d - h$. We consider what effect this has on the overall flow in the system, as well as the flow on each of the other player's links. The total flow in the the system increases by an amount not more than $h\frac{1}{s}$ because of the concavity assumption. Let $\frac{q(h)}{s} \le \frac{h}{s}$ be the actual amount of the increase, and note that $\lim_{h\to 0} \frac{q(h)}{h} \to 1$. Similarly, the flow in each other link j decreases by $h\frac{1}{a_j}$, provided that the original flow on link j was at least $h\frac{1}{a_j}$. To cover the possibility that the original flow were not this large define $y_j(h) = \min(f_j a_j, h)$. Then, $\lim_{h\to 0} \frac{y_j(h)}{h} = 1$ and the flow in link j decreases by $\frac{y_j(h)}{a_j}$. Thus the flow on link i increases by $\sum_{j\neq i} \frac{y_j(h)}{a_j} + \frac{q(h)}{s}$. For notational convenience, let

$$\theta(h) = \left[\sum_j \frac{y_j(h)/h}{a_j} + \frac{q(h)/h}{s}\right].$$

The flow increase on link i can now be expressed simply as $h\theta$. The new price p_i'can be found by taking the difference between the new disutility and the new latency on i's link. This difference is

$$\begin{aligned} p_i' &= (d-h) - a_i\left(f_i + \sum_{j\neq i}\frac{y_j(h)}{a_j} + \frac{q(h)}{s}\right) - b_i \\ &= p_i - h - a_i h\,\theta(h) \end{aligned}$$

where p_i is the original price. We write an expression for the new profit π' by taking the product of the new price and flow which is

$$\begin{aligned} &\pi' = (p_i - h - a_i h\theta(h))\,(f_i + h\theta(h)) = \\ &-h^2(1 + a_i\theta(h))\theta(h) + h\left[(p_i - a_i f_i)\,\theta(h) - f_i\right] + \pi. \end{aligned}$$

We would like to find conditions for π' is not greater than the old profit π for any h. The first order condition requires that the linear term of the above quadratic form in h have a coefficient of 0. This in turn requires that

$$f_i = \frac{p_i(\sum_{j\neq i}\frac{1}{a_j}+\frac{1}{s})}{1+a_i(\sum_{j\neq i}\frac{1}{a_j}+\frac{1}{s})}.$$

Note that this condition is also sufficient to insure that $h = 0$ is a global maximum, because under this condition, the linear term in h vanishes and all that is left is the term that depends on h^2 that has a negative coefficient. Reducing the condition further, we find that the condition is equivalent to

$$\frac{p_i}{f_i} = a_i + \left(\sum_{j\neq i}\frac{1}{a_j}+\frac{1}{s}\right)^{-1}. \tag{2}$$

■

Now consider the socially optimal pricing. We first give an intuitive argument why the socially optimal price should be $a_i f_i^*$ where f_i^* is the flow of link i in social optimum. The socially optimal pricing should price the flow so that each user bears the marginal cost to society of each additional unit of new flow. The latency on each link is $a_i f_i^* + b_i$. However, the social cost is the latency times the amount of flow bearing that latency. Therefore the cost is $a_i f_i^{*2} + b_i f_i^*$. Thus the marginal cost is $2a_i f_i^* + b_i$. The latency born by users is $a_i f_i^* + b_i$. Therefore to make the disutility born by users reflect marginal social cost, the price should be set to $a_i f_i^*$.

We formalize the argument in the lemma and proof that follows.

Lemma 3: In the game $\mathcal{G}$ the social optimum price vector satisfies

$$p_i = a_i f_i^*$$

for all i where f_i^* is the flow in social optimum.

Proof: If a set of prices were to induce a flow vector that maximizes social welfare, that set of prices would be socially optimal. One way to quantify social welfare is to integrate the area under the disutility curve $U(x)$ up to the amount of flow carried, and then subtract the welfare lost due to the latency in each link. Therefore the socially optimal flow can be found by solving the following optimization problem:

$$\begin{aligned} \max \quad & \left[\int_0^f U(x)\right] - \sum_i (a_i f_i^2 + b_i f_i) \\ \text{s.t.} \quad & \sum_i f_i - f = 0, \\ & f_i \geq 0\,\forall i. \end{aligned}$$

Note that the function $-\left[\int_0^f U(x)\right] + \sum_i (a_i f_i^2 + b_i f_i)$ is convex and Slater's constraint qualification condition holds so there is no duality gap if we use Lagrangian techniques to find the optimal solution [9]. We therefore may express the solution to the above problem by writing the Lagrangian, evaluating the first order conditions as well as the complementary slackness conditions for the Lagrange multipliers associated with the inequality constraints and simplifying. This yields

$$\begin{cases} d^* - 2a_i f_i^* - b_i = 0 & f_i^* > 0 \\ d^* - b_i < 0 & f_i^* = 0 \end{cases}$$

where $d^* = U(f^*)$ is the disutility of the used links in the optimal solution. In our model the difference between the disutility and the link latency should be the price of the link. The above expression shows that for each used link i, the disutility minus the latency $a_i f_i^* + b_i$ is equal to $a_i f_i^*$, therefore the prices that achieve a socially optimal flow are $a_i f_i^*$. ■

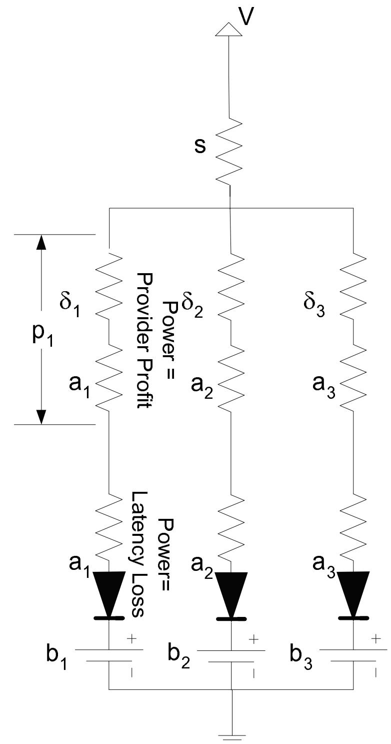

Fig. 1. A circuit analogous to the network pricing game.

The proof of Lemma 3 also demonstrates that the optimal price achieves the optimal flow vector. In other words, if we were to give the social planner the power to assign user routes (choose the flow vector) the planner could not achieve a better welfare than by merely choosing the link prices.

B. Circuit Analogy

Before we make the analogy between the game and a network of resistors, we introduce an illustration like the one used in [2] to visualize the relations between flow f_i, price p_i, latency function slope a_i and offset b_i for each link i. The illustration is shown in panel (i) of Figure 2. The figure shows that the price p_i plus latency $a_i f_i + b_i$ of each link is equal to the common value, $U(f_1 + f_2 + f_3) = U(f)$. The figure also shows the areas that correspond to link owner profit and user surplus.

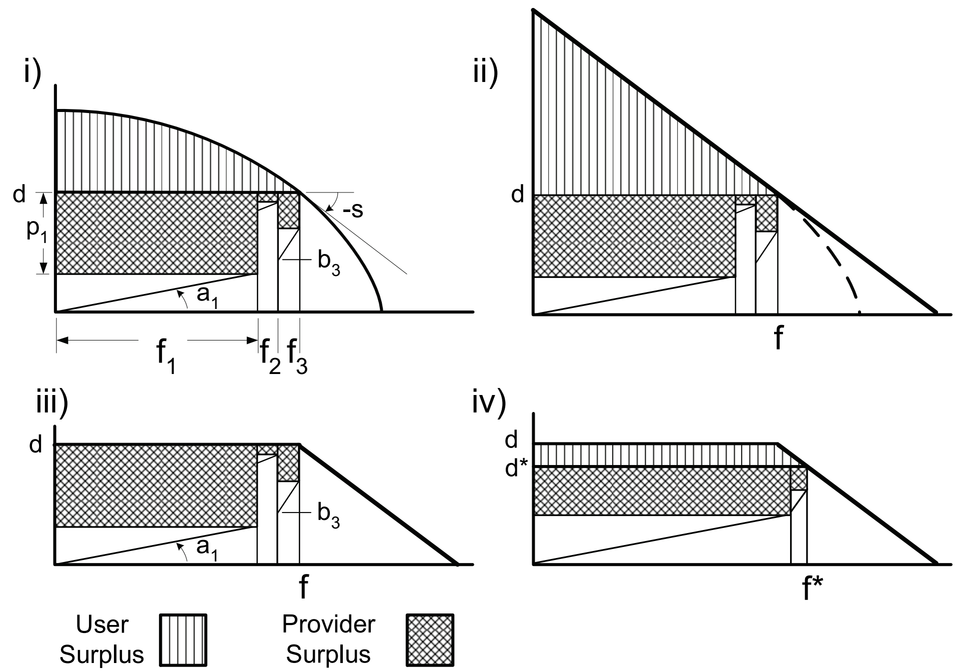

Fig. 2. i) The Nash equilibrium of the game $\mathcal{G}$. ii) The Nash equilibrium of the game $\mathcal{G}_l$, where the disutility function of $\mathcal{G}$ has been linearized. Note that the flow vector is unchanged from the Nash equilibrium of $\mathcal{G}$. iii) The Nash equilibrium of the game $\mathcal{G}_t$, where the disutility function of $\mathcal{G}$ has been linearized and "truncated." iv) The social optimum configuration of the game $\mathcal{G}_t$. Note that the flow $f^* > f$ and that link 2 is not used in the social optimum configuration of this example.

We illustrate our analogy to a network of resistors in Figure 1. In the analogy, each link is one of n parallel branches in a circuit. The current in the branch i is analogous to the flow on link i and the latency is analogous to the voltage drop across a resistor of a_i ohms and voltage source of b_i volts connected in series. To ensure that our imaginary voltage source does not ever cause flow to go the wrong way (become negative), we place a diode in each branch of the circuit.

The price p_i is found by taking the voltage drop across a resistor of $\delta_i + a_i$ ohms where

$$\delta_i = \left(\sum_{j \neq i} \frac{1}{a_j} + \frac{1}{s} \right)^{-1} = \frac{p_i}{f_i} - a_i. \tag{3}$$

The price plus the latency on a link equals the common disutility found across all used links. In the circuit analogy, the voltage representing price plus the voltage representing latency equals the common voltage d across all the parallel branches where

$$d = p_i + a_i f_i + b_i = (2a_i + \delta_i) f_i + b_i.$$

The value d should also correspond to the point on the value $U(f)$, and there is a circuit analogy for this correspondence as well. Suppose we let V be the y-intercept of the tangent to the disutility curve at the point $U(f) = d$. Then

$$V - s \sum f_i = d.$$

This is analogous to having a battery of V volts with an internal resistance of s ohms. As the current drawn from the battery increases, the voltage output (analogous to the willingness to tolerate disutility) decreases.

Alternatively, we could avoid linearizing by considering a general power source with a nonlinear current to voltage characteristic. In particular the current to voltage function should matches the disutility function $U(\cdot)$ which of course relates flow to disutility.

To model the social optimum configuration, we keep the general power source but replace the resistor modelling the price-to-flow relationship from $a_i + \delta_i$ to a_i on each branch i. Clearly, by decreasing the resistance in all branches, the total amount of current (flow) should increase, and the voltage drop (disutility d^*) on all branches should decrease. However, the ratio of the flows between branches can change, because the ratio of the resistances changes between the two cases. Therefore it is possible that the flow on a particular branch decreases in social optimum. The flow on a branch can even become zero if $d^* \leq b_i$. We also note that a branch with a low enough b_i to carry a nonzero flow in the social optimum configuration should also carry a nonzero flow in the Nash equilibrium configuration. In other words, a link that is used in social optimum should be used in Nash equilibrium.

Intuition suggests that if $\delta_i << a_i$ were small for all i, the difference between social optimum and Nash equilibrium should also be small. Indeed δ_i is smaller than a_j for all $j \neq i$. However, for the smallest a_i, it could be that $\delta_i > a_i$.

Thus we cannot attempt to show that δ_i is negligibly small. In the next section, we will describe the relation between the δ_i's and a_i's, using linear algebra in order to develop our proof on the price of anarchy bound.

C. The Price of Anarchy for Linear Latency

In this section we will show that the price of anarchy is at most 1.5 for linear latency functions, where price of anarchy is defined to be the ratio of welfare in social optimum to the welfare in Nash equilibrium. We will first argue that we can restrict our attention to examples where the disutility function has been "linearized" for values larger or equal to the equilibrium flow f and has been "truncated" or set to be equal a constant for smaller values of flow. This argument is also made in [2], but presented again here for completeness. We illustrate the progression from the original game, to a game with a linearized disutility function, and finally to a truncated disutility function in Figure 2. Once we have shown that we may restrict our analysis to truncated linear disutility functions, we use techniques of linear algebra to prove the final result.

Before we begin to prove the 1.5 bound of the price of anarchy, we give an example where the price of anarchy is exactly 1.5. Consider a network consisting of only one link where the disutility function is

$$U(f) = \begin{cases} 1 & f \leq 0 \\ 2 - f & 1 \leq f \leq 2. \end{cases}$$

and the latency function is $l(f) = 0$. It is easy to verify that the Nash equilibrium price is $p = 1$ which yields a profit of 1 for the provider and 0 user surplus. The socially optimal price is $p = 0$ which yields a profit of 0 for the provider, but a user surplus of $3/2$. Therefore the price of anarchy in this example is 1.5. The rest of this section will prove that there is not an example where the price of anarchy is larger than this.

Lemma 4: Let f be the total Nash equilibrium flow of the game $\mathcal{G}$. Let $\mathcal{G}_l$ be a new game identical to $\mathcal{G}$ except that the disutility function of the new game is a line tangential to the disutility function of $\mathcal{G}$ at the point $U(f) = d$. Then both the Nash equilibrium flow and price vectors of the new game $\mathcal{G}_l$ are the same as the Nash flow and price vectors of the old game $\mathcal{G}$.

Proof: The Nash flow vector of $\mathcal{G}$ satisfies the necessary and sufficient condition in Lemma 2 to be a Nash equilibrium. Because the slope of the disutility function of $\mathcal{G}_l$ is the same when the total flow is f, the Lemma 2 conditions for $\mathcal{G}_l$ evaluated at the Nash flow vector of $\mathcal{G}$ are satisfied. Thus the Nash flow vector of $\mathcal{G}$ is also a Nash flow for $\mathcal{G}_l$ ■

Lemma 5: Consider the game $\mathcal{G}_t$ that consists of the same latency functions as $\mathcal{G}$ and $\mathcal{G}_l$ but has a disutility curve that is modified from the disutility curve in $\mathcal{G}_l$ by "truncating" it in the following way:

$$U(x) = \begin{cases} d & x \leq f \\ d - s(x - f) & x > f. \end{cases}$$

where $s = U'(f)$. Then the Nash equilibrium flow and price vectors of the new game $\mathcal{G}_t$ are the same as the Nash flow and price vectors of the original game $\mathcal{G}$.

Proof: The argument is essentially the same as presented in [2]. Recall that Lemma 2 found that relation (2) holds in Nash equilibrium whenever the disutility function is concave and everywhere differentiable. To prove this result, we must show that (2) still holds for the truncated disutility function. While the truncated disutility function is no longer both-sided differentiable at the equilibrium point, it is right hand differentiable. The first order condition on flow f_i found with the right hand derivative is almost the same as was found in Lemma 2 but instead it holds with inequality. More precisely

$$f_i \geq \frac{p_i(\sum_{j \neq i} \frac{1}{a_j} + \frac{1}{s})}{1 + a_i(\sum_{j \neq i} \frac{1}{a_j} + \frac{1}{s})}$$

for all i is the necessary and sufficient test to identify a Nash equilibrium of game $\mathcal{G}_t$. However, the Nash equilibrium flow vector for game $\mathcal{G}$, must satisfy the above condition with equality, therefore it is also a Nash equilibrium of $\mathcal{G}_t$. ■

Lemma 6: Let $N(\mathcal{G})$ and $S(\mathcal{G})$ be the social welfare in the Nash equilibrium and social optimum configurations of the game $\mathcal{G}$. Then

$$\frac{S(\mathcal{G}_t)}{N(\mathcal{G}_t)} > \frac{S(\mathcal{G})}{N(\mathcal{G})}.$$

Proof: Let $U(\mathcal{G})$ be the user welfare in the Nash equilibrium of game $\mathcal{G}$. In the Nash equilibrium of game $\mathcal{G}_t$, Lemma 6 shows that the flow vector will be the same as the Nash flow of $\mathcal{G}$. Thus the provider welfare will be the same in both cases. However, the user welfare will be 0. Thus we have

$$N(\mathcal{G}_t) = N(\mathcal{G}) - U(\mathcal{G}).$$

Consider a new game $\mathcal{G}_{tx}$ with the same latency functions as $\mathcal{G}$ but with a disutility function that has been truncated but not linearized. Therefore

$$U_{tx}(x) = \begin{cases} d & x \leq f \\ U(x) & x > f. \end{cases}$$

where d and f are the Nash equilibrium disutility and flow of $\mathcal{G}$, $U_{tx}(\cdot)$ is the disutility function of $\mathcal{G}_{tx}$, and $U(\cdot)$ is the disutility function of $\mathcal{G}$. Because we know the disutility d^* in social optimum is less than d we can view the optimization problem of finding the optimal flow vector for game $\mathcal{G}_{tx}$ as the same problem as finding the optimal flow vector for the game $\mathcal{G}$ but with the constant $U(\mathcal{G}) = \int_0^f (U(x) - d)dx$ subtracted off the objective function. Thus, the social optimum flow vector will be identical for game $\mathcal{G}_{tx}$ as for game $\mathcal{G}_t$. Thus

$$S(\mathcal{G}_{tx}) = S(\mathcal{G}) - U(\mathcal{G}).$$

Now consider the game $\mathcal{G}_t$ where we have both truncated and linearized the disutility function. By convexity, the disutility function of $\mathcal{G}_t$ is never smaller than that of $\mathcal{G}_{tx}$. Thus the solution to the optimization problem of finding the optimal flow vector has an objective function that is never smaller than the objective function that we maximized to find $S(\mathcal{G}_{tx})$. Consequently

$$S(\mathcal{G}_t) \geq S(\mathcal{G}_{tx}) = S(\mathcal{G}) - U(\mathcal{G}).$$

Combining our observations, we have

$$\frac{S(\mathcal{G}_t)}{N(\mathcal{G}_t)} \geq \frac{S(\mathcal{G}) - U(\mathcal{G})}{N(\mathcal{G}) - U(\mathcal{G})} \geq \frac{S(\mathcal{G})}{N(\mathcal{G})}$$

■

We define the following notation. The vectors of flows in Nash equilibrium and social optimum are

$$F = [f_1, f_2, ..., f_n]^T \text{ , and } F^* = [f_1^*, f_2^*, ..., f_n^*]^T$$

respectively. We define

$$\mathbf{V} = [V, V, ...V]^T \text{ , and } b = [b_1, b_2, ...b_n]^T$$

as an n dimensional vector of all V's and the vector of b_i's respectively. The matrices

$$A = \text{diag}(a_1, a_2, ..., a_n) \text{ , and } \Delta = \text{diag}(\delta_1, \delta_2, ..., \delta_n)$$

are diagonal matrices of the $a_i's$ and δ_i's respectively where recall δ_i is defined by 3. It will also be convenient to define

$$M = \begin{bmatrix} 1 & 1 & ... \\ 1 & 1 & ... \\ \vdots & \vdots & \ddots \end{bmatrix}$$

to be a $n \times n$ matrix of all ones.

Without loss of generality, we may renumber the links so that links used both in Nash equilibrium and in social optimum are numbered $1...m$ and the links used in Nash equilibrium but not social optimum are numbered $(m + 1)...(m + k) = n$. There cannot exist links used in social optimum that are not used in Nash equilibrium, while there could be links that are not used in social optimum nor in Nash equilibrium, but we will only consider examples where such links are removed. Thus, we may define $\bar{A}$, $\bar{\Delta}$, be the upper $m \times m$ blocks of the matrices A and Δ respectively. These blocks contain the a_i's and δ's associated with links that used both in Nash equilibrium and social optimum.

Similarly we define $\underline{A}$, $\underline{\Delta}$ to be the lower $k \times k$ blocks of A and Δ respectively. These blocks contain the a_i's and δ's associated with links that are used both in Nash equilibrium but are undercut (not used) in social optimum. We therefore have

$$A = \begin{bmatrix} \bar{A}, & 0 \\ 0, & \underline{A} \end{bmatrix} \text{ and } \Delta = \begin{bmatrix} \bar{\Delta}, & 0 \\ 0, & \underline{\Delta} \end{bmatrix}.$$

Theorem 1: Consider the pricing game $\mathcal{G}_t$ with linear latency functions, and a truncated linear disutility function. The difference between three times the Nash welfare $W_t = N(\mathcal{G}_t)$ and two times the social welfare $W_t^* = S(\mathcal{G}_t)$ can be expressed as

$$3W_t - 2W_t^* = [\bar{F}^T \underline{F}^T] \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} \bar{F} \\ \underline{F} \end{bmatrix} \qquad (4)$$

where

$$\begin{aligned} H_{11} &= \bar{A} + \bar{\Delta} - \bar{\Delta}(2\bar{A} + sM)^{-1}\bar{\Delta}, \\ H_{12} &= -s\bar{\Delta}(2\bar{A} + sM)^{-1}1_{m\times k}, \\ H_{21} &= H_{12}^T, \text{ and} \\ H_{22} &= 3\underline{A} + 3\underline{\Delta} + s1_{k\times k} - s^2 1_{k\times m}(2\bar{A} + sM)^{-1}1_{m\times k}. \end{aligned}$$

Proof: We have that

$$(2A + sM + \Delta)F = \mathbf{V} - b. \qquad (5)$$

We can write a similar expression for the flow in social optimum case, with the following modification. In social optimum, the disutility d^* may fall below b_i for some i, so that some links that were used in Nash equilibrium may become not used in the social optimum case. Without loss of generality, we renumber the links so that the the used links number $1...m$ and the unused links number $(m+1)...(m+k) = n$. Let $\bar{A}$, $\bar{M}$, $\bar{\Delta}$, be the upper m by m blocks of the matrices A, M and Δ respectively. We therefore have

$$[2\bar{A} + s\bar{M}, 0_{m\times k}]F^* = [I_{m\times m}, 0_{m\times k}](\mathbf{V} - b)$$

or equivalently

$$F^* = \begin{bmatrix} (2\bar{A} + s\bar{M})^{-1} & 0_{m\times k} \\ 0_{k\times m} & 0_{k\times k} \end{bmatrix} (\mathbf{V} - b).$$

By substituting (5) we find

$$F^* = \begin{bmatrix} (2\bar{A} + s\bar{M})^{-1} & 0_{m\times k} \\ 0_{k\times m} & 0_{k\times k} \end{bmatrix} \times \begin{bmatrix} 2\bar{A} + s\bar{M} + \bar{\Delta} & s1_{m\times k} \\ s1_{k\times m} & 2\underline{A} + s\underline{M} + \underline{\Delta} \end{bmatrix} F$$

which reduces to

$$F^* = \begin{bmatrix} I + (2\bar{A} + s\bar{M})^{-1}\bar{\Delta} & s(2\bar{A} + s\bar{M})^{-1}1_{m\times k} \\ 0_{k\times m} & 0_{k\times k} \end{bmatrix} F.$$

With a linear (not truncated) disutility curve, the total social welfare gathered in Nash equilibrium is given by

$$W = F^T(A + \frac{1}{2}sM + \Delta)F$$

while the welfare gathered by the users is $\frac{s}{2}F^T MF$. Therefore the Nash equilibrium social welfare for a truncated linear demand curve is

$$W_t = F^T(A + \Delta)F. \qquad (6)$$

For a linear (not truncated) disutility function the social welfare gathered in the optimal configuration is given by

$$W^* = F^{*T}(A + \frac{1}{2}sM)F^*$$

While for a linear truncated disutility function, the social welfare in the optimal configuration is

$$\begin{aligned} W_t^* &= \frac{1}{2}F^{*T}(2\bar{A} + sM)F^* - F^T(\frac{1}{2}sM)F \\ &= \frac{1}{2}\bar{F}^{*T}\left[(2\bar{A} + sM) + \bar{\Delta}, \quad s1_{m\times k}\right]F - \frac{1}{2}F^T(sM)F \\ &= \frac{1}{2}F^T \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} F \end{aligned} \qquad (7)$$

where

$$\begin{aligned}
G_{11} &= 2\bar{A} + 2\bar{\Delta} + \bar{\Delta}(2\bar{A} + sM)^{-1}\bar{\Delta}, \\
G_{12} &= s\bar{\Delta}(2\bar{A} + sM)^{-1}1_{m\times k}, \\
G_{21} &= G_{12}^T, \text{ and} \\
G_{22} &= s^2 1_{k\times m}(2\bar{A} + sM)^{-1}1_{m\times k} - s1_{k\times k}.
\end{aligned}$$

The Nash equilibrium social welfare, taken from expression (6) but expressed using the block matrix notation is

$$W_t = F^T \begin{bmatrix} \bar{A} + \bar{\Delta} & 0 \\ 0 & \underline{A} + \underline{\Delta} \end{bmatrix} F. \tag{8}$$

We can use expressions (7) and (8) to express three times the Nash welfare minus two times the social optimal welfare as

$$3W_t - 2W_t^* = [\bar{F}^T \underline{F}^T] \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} \bar{F} \\ \underline{F} \end{bmatrix}$$

where

$$\begin{aligned}
H_{11} &= \bar{A} + \bar{\Delta} - \bar{\Delta}(2\bar{A} + sM)^{-1}\bar{\Delta}, \\
H_{12} &= -s\bar{\Delta}(2\bar{A} + sM)^{-1}1_{m\times k}, \\
H_{21} &= H_{12}^T, \text{ and} \\
H_{22} &= 3\underline{A} + 3\underline{\Delta} + s1_{k\times k} - s^2 1_{k\times m}(2\bar{A} + sM)^{-1}1_{m\times k}.
\end{aligned}$$

■

Lemma 7: Define

$$\begin{aligned}
\beta &\triangleq 2/s + \mathrm{tr}(\bar{A}^{-1}), \\
\alpha &\triangleq 1/s + \mathrm{tr}(A^{-1}), \text{ and} \\
e &\triangleq 1_{m\times 1}.
\end{aligned}$$

Provided that $a_i > 0$ for all i, the following identities are true:

$$\begin{aligned}
(s\bar{A} + s\bar{M})^{-1} &= \frac{1}{2}\left[\bar{A}^{-1} - \frac{1}{\beta}\bar{A}^{-1}M\bar{A}^{-1}\right] \\
\bar{\Delta} &= (\alpha\bar{A} - I)^{-1}\bar{A} \\
\bar{\Delta}(2\bar{A} + sM)^{-1}e &= \frac{1}{s\beta}(\alpha\bar{A} - I)^{-1}e \\
e^T(2\bar{A} + s\bar{M})^{-1}e &= \frac{\mathrm{tr}(A^{-1})}{s\beta}
\end{aligned}$$

Proof: The Matrix Inversion Lemma states that

$$(F + UCV)^{-1} = F^{-1} - F^{-1}U(C^{-1} + VF^{-1}U)^{-1}VF^{-1}$$

where F, U, C, V are arbitrary matrices with only the condition that the matrices are of appropriate dimension and all the inverses in the above expression exist. Let $F = 2\bar{A}$, $U = e$, $V = e^T$, $C = s/2$ and note that $M = ee^T = 1_{m\times m}$. Then we may apply the Matrix Inversion Lemma in the following way:

$$\begin{aligned}
(2\bar{A} + s\bar{M})^{-1} &= \frac{1}{2}\bar{A}^{-1} - \frac{1}{4}\bar{A}^{-1}e\left[\frac{1}{s} + \frac{e^T\bar{A}^{-1}e}{2}\right]^{-1} e^T\bar{A}^{-1} \\
&= \frac{1}{2}\left[\bar{A}^{-1} - \bar{A}^{-1}e\left[\frac{2}{s} + \mathrm{tr}(\bar{A}^{-1})\right]^{-1} e^T\bar{A}^{-1}\right] \\
&= \frac{1}{2}\left[\bar{A}^{-1} - \frac{1}{\beta}\bar{A}^{-1}M\bar{A}^{-1}\right].
\end{aligned}$$

To derive the next identity, recall $\delta_i = \left[\sum_{j\neq i}\frac{1}{a_i} + \frac{1}{s}\right]^{-1}$. Therefore

$$\begin{aligned}
\bar{\Delta}^{-1} &= \left[\mathrm{tr}(A^{-1}) + \frac{1}{s}\right] - \bar{A}^{-1} \\
&= \alpha I - \bar{A}^{-1}.
\end{aligned}$$

Thus $\bar{A}\bar{\Delta}^{-1} = \alpha\bar{A} - I$, and therefore $\bar{\Delta}\bar{A}^{-1} = (\alpha\bar{A} - I)^{-1}$. Note by construction, $\alpha > 1/\min_i a_i$ so that $(\alpha\bar{A} - I)^{-1}$ exists. Finally we may conclude that $\bar{\Delta} = (\alpha\bar{A} - I)^{-1}\bar{A}$.

To derive the next identity we observe that:

$$\begin{aligned}
&\bar{\Delta}(2\bar{A} + sM)^{-1}e \\
&\quad = \frac{1}{2}(\alpha\bar{A} - I)^{-1}e - \frac{1}{2\beta}[\alpha\bar{A} - I]^{-1}ee^T\bar{A}^{-1}e \\
&\quad = \frac{1}{2}\left[(\alpha\bar{A} - I)^{-1}e - \frac{\mathrm{tr}(\bar{A}^{-1})}{\mathrm{tr}(\bar{A}^{-1}) + 2/s}(\alpha\bar{A} - I)^{-1}e\right] \\
&\quad = \frac{1}{s\beta}(\alpha\bar{A} - I)^{-1}e.
\end{aligned}$$

The final identity follows from the following reasoning:

$$\begin{aligned}
e^T(2\bar{A} + s\bar{M})^{-1}e &= \frac{1}{2}\left[e^T\bar{A}^{-1}e - \frac{1}{\beta}e^T\bar{A}^{-1}ee^T\bar{A}^{-1}e\right] \\
&= \frac{1}{2}\left[\mathrm{tr}(A^{-1}) - \frac{\mathrm{tr}(A^{-1})^2}{\mathrm{tr}(A^{-1}) + 2/s}\right] \\
&= \frac{\mathrm{tr}(A^{-1})}{s\beta}.
\end{aligned}$$

■

With the identities of Lemma 7, we can now prove the main result of the section.

Theorem 2: Consider the pricing game $\mathcal{G}$ with affine latency functions. The price of anarchy, defined as $\frac{S(\mathcal{G})}{N(\mathcal{G})}$ is at most 1.5.

Proof: By Lemma 6 it is sufficient to show that the price of anarchy for a game with a truncated linear disutility function is 1.5. Without loss of generality, we may assume that all links are used in Nash equilibrium, because if otherwise, a link not used in Nash equilibrium would not be used in Social Optimum. Therefore one can discard the links unused in Nash equilibrium from the game without changing the price of anarchy.

Applying the identities in Lemma 7 to the bound of Theorem 1, we have

$$\begin{aligned}
H_{11} &= (\alpha\bar{A} - I)^{-1}(\alpha^2\bar{A}^3 - \alpha\bar{A}^2 - \frac{1}{2}\bar{A} + \frac{1}{2\beta}\bar{M})(\alpha\bar{A} - I)^{-1} \\
H_{12} &= \frac{1}{s\beta}(\alpha\bar{A} - I)^{-1}e_m e_k \\
H_{21} &= H_{12}^T \\
H_{22} &\geq s1_{k\times k} - s^2 e_k e_m^T(2\bar{A} + s\bar{M})^{-1}e_m e_k^T \\
&= \frac{2 + s\,\mathrm{tr}(\bar{A}^{-1})}{\beta}1_{k\times k} - \frac{s\,\mathrm{tr}(\bar{A}^{-1})}{\beta}1_{k\times k} \\
&= \frac{2}{\beta}1_{k\times k}
\end{aligned}$$

where recall that H_{11}, H_{12}, H_{21}, and H_{22} are the blocks of the matric in expression (4).

Let

$$Q \triangleq (\alpha\bar{A} - I)^{-1}\bar{F} \text{ and}$$
$$Z \triangleq ||\underline{F}||_1 = e_k\underline{F}.$$

Note that $Q > 0$ because $\alpha a_i > 1$ for all i. By changing variables from $\bar{F}$ and $\underline{F}$ to Q and Z we obtain that

$$\begin{aligned}
&3W_t - 2W_t^* \\
&\geq [Q^T, Z^T]\begin{bmatrix}\alpha^2\bar{A}^3 - \alpha\bar{A}^2 - \frac{1}{2}\bar{A} + \frac{1}{2\beta}\bar{M} & -\frac{1}{\beta}e_m \\ -\frac{1}{\beta}e_m^T & \frac{2}{\beta}\end{bmatrix}\begin{bmatrix}Q \\ Z\end{bmatrix} \\
&= \frac{1}{2\beta}||Q||_1^2 - \frac{2}{\beta}||Q||_1 Z + \frac{2}{\beta}Z^2 + \\
&\qquad Q^T(\alpha^2\bar{A}^3 - \alpha\bar{A}^2 - \frac{1}{2}\bar{A})Q. \qquad (9)
\end{aligned}$$

Note that the quadratic $|Q||_1^2 - 2||Q||_1 Z + 2Z^2$ is nonnegative. Consider the following two cases.

Case 1: Suppose $\alpha^2\bar{A}^3 - \alpha\bar{A}^2 - \frac{1}{2}\bar{A} \geq 0$, or equivalently $\alpha\bar{a}_j \geq \frac{1+\sqrt{3}}{2}$ for each $\bar{a}_j$ on the diagonal of $\bar{A}$. Then $3W_t - 2W_t^* > 0$.

Case 2: Suppose that for some j, $\alpha\bar{a}_j < \frac{1+\sqrt{3}}{2}$. Let $x \triangleq \alpha\bar{a}_j - 1$. Let j be the index of any such $\bar{a}_j$

We therefore have

$$\alpha\bar{a}_j = x + 1 < \frac{1+\sqrt{3}}{2}, \qquad (10)$$

We divide both sides of the equality by $\bar{a}_j$, and substitute that $\alpha \triangleq \text{tr}(A^{-1}) + \frac{1}{s} = \text{tr}(\bar{A}^{-1}) + \text{tr}(\underline{A}^{-1}) + \frac{1}{s}$ to obtain that

$$\frac{x+1}{\bar{a}_j} = \sum_i \frac{1}{\bar{a}_i} + \sum_i \frac{1}{\underline{a}_i} + \frac{1}{s}$$

where $\{\underline{a}_i\}$ are the diagonal entries of $\underline{A}$. Thus

$$\frac{x}{\bar{a}_j} = \sum_{i \neq j} \frac{1}{\bar{a}_i} + \sum_i \frac{1}{\underline{a}_i} + \frac{1}{s}.$$

Define

$$\bar{a}_{-j} \triangleq \left[\sum_{i\neq j}\frac{1}{\bar{a}_i}\right]^{-1} \qquad \underline{a} \triangleq \left[\sum_i \frac{1}{\underline{a}_i}\right]^{-1}$$

To extend our circuit analogy, $\bar{a}_{-j}$ is the equivalent resistance of all of the resistors $\{\bar{a}_i\}_{i\neq j}$ connected in parallel while $\underline{a}$ is the equivalent resistance of the resistors $\{\underline{a}_i\}$ connected in parallel. After substituting these definitions, we have that

$$\frac{x}{\bar{a}_j} = \frac{1}{\bar{a}_{-j}} + \frac{1}{\underline{a}} + \frac{1}{s}$$

and thus

$$\bar{a}_j = x\left[\frac{1}{\bar{a}_{-j}} + \frac{1}{\underline{a}} + \frac{1}{s}\right]^{-1}. \qquad (11)$$

Recall that

$$\delta_j = \left[\frac{1}{\bar{a}_{-j}} + \frac{1}{\underline{a}} + \frac{1}{s}\right]^{-1}.$$

From these last two relations, we have that

$$(2\bar{a}_j + \delta_j) = (2x+1)\left[\frac{1}{\bar{a}_{-j}} + \frac{1}{\underline{a}} + \frac{1}{s}\right]^{-1}.$$

We will now find an upper bound on the total Nash equilibrium flow on links that are not used in social optimum. Thus

$$\begin{aligned}
||\underline{F}||_1 = \sum \frac{d - \underline{b}_i}{2\underline{a}_i + \underline{\delta}_i} &< [d - \min_i(\underline{b}_i)]\sum\frac{1}{2\underline{a}_i} \\
&= \frac{1}{2\underline{a}}[d - \min_i(\underline{b}_i)].
\end{aligned}$$

We also note that

$$\bar{F}_j = \frac{d - \bar{b}_j}{2\bar{a}_j + \bar{\delta}_j}.$$

Consequently,

$$\frac{||\underline{F}||_1}{\bar{F}_j} \leq \frac{[d - \min_i(\underline{b}_i)]}{d - \bar{b}_j}\frac{2\bar{a}_j + \bar{\delta}_j}{2\underline{a}}.$$

We note that $\underline{b}_i > \bar{b}_j$ for any i because links that are used in social optimum must have a lower 'b' than links that are not used in social optimum. Therefore

$$\begin{aligned}
\frac{||\underline{F}||_1}{\bar{F}_j} &\leq \frac{2\bar{a}_j + \bar{\delta}_j}{2\underline{a}} \\
&= \frac{2x+1}{2\underline{a}}\left[\frac{1}{\bar{a}_{-j}} + \frac{1}{\underline{a}} + \frac{1}{s}\right]^{-1}.
\end{aligned}$$

We also observe that

$$||Q||_1 = e^T(\alpha\bar{A} - I)^{-1}\bar{F} = \sum\frac{F_i}{\alpha\bar{a}_i - 1} \geq \frac{F_j}{\alpha\bar{a}_j - 1} = \frac{F_j}{x}.$$

Thus if we recall $Z \triangleq ||\underline{F}||_1$, we obtain

$$\frac{Z}{||Q||_1} \leq \frac{(2x+1)x}{2\underline{a}}\left[\frac{1}{\bar{a}_{-j}} + \frac{1}{\underline{a}} + \frac{1}{s}\right]^{-1}.$$

Let

$$\upsilon = \frac{(2x+1)x}{2\underline{a}}\left[\frac{1}{\bar{a}_{-j}} + \frac{1}{\underline{a}} + \frac{1}{s}\right]^{-1}.$$

Then $Z < \upsilon||Q||_1$. If $Z \in [0, \frac{1}{2}||Q||_1]$ then $|Q||_1^2 - 2||Q||_1 Z + 2Z^2$ decreases with Z. From the expression above we see that $\Upsilon < \frac{1}{2}(2x+1)x$, while we also have that $x < \frac{\sqrt{3}-1}{2} < 0.367$ from (10). This implies $\upsilon < .32 < .5$ and therefore Z must lie in the range where $|Q||_1^2 - 2||Q||_1 Z + 2Z^2$ decreases in Z. Thus we have

$$\begin{aligned}
|Q||_1^2 - 2||Q||_1 Z + 2Z^2 &> |Q||_1^2(1 - 2\upsilon + 2\upsilon^2) \\
&= Q^T\left((1 - 2\upsilon + 2\upsilon^2)M\right)Q.
\end{aligned}$$

After we substitute the above bound into (9) we obtain

$$\begin{aligned}
&3W_t - 2W^* > \\
&\quad Q^T((\frac{2}{\beta}\upsilon^2 - \frac{2}{\beta}\upsilon + \frac{1}{2\beta})M + \alpha^2\bar{A}^3 - \alpha\bar{A}^2 - \frac{1}{2}\bar{A})Q.
\end{aligned}$$

We define Λ to be the matrix between Q^T and Q in the above expression. Therefore

$$\Lambda \triangleq ((\frac{2}{\beta}\upsilon^2 - \frac{2}{\beta}\upsilon + \frac{1}{2\beta})M + \alpha^2 \bar{A}^3 - \alpha\bar{A}^2 - \frac{1}{2}\bar{A}).$$

Note that it is sufficient to show that all entries of Λ are non-negative because Q is non-negative. The off-diagonal elements of Λ are clearly positive. Any diagonal element k satisfying $\alpha\bar{a}_k \geq \frac{1+\sqrt{3}}{2}$ is also non-negative because this condition makes $\alpha^2\bar{a}_k^3 - \alpha\bar{a}_k^2 - \frac{1}{2}\bar{a}_k$ non-negative and $(\frac{2}{\beta}\upsilon^2 - \frac{2}{\beta}\upsilon + \frac{1}{2\beta})$ is always nonneagtive. Thus we can focus on diagonal elements of index j' satisfying $\alpha\bar{a}_{j'} < \frac{1+\sqrt{3}}{2}$ and recall we defined index j to be any arbitrary index for which $\alpha\bar{a}_{j'} < \frac{1+\sqrt{3}}{2}$ is satisfied. Thus it is sufficient to show that jth diagonal element which is

$$\lambda_{jj} = \frac{2}{\beta}\upsilon^2 - \frac{2}{\beta}\upsilon + \frac{1}{2\beta} + \alpha^2\bar{a}_j^3 - \alpha\bar{a}_j^2 - \frac{1}{2}\bar{a}_j. \qquad (12)$$

is nonnegative . Equivalently, we need to show that

$$\lambda_{jj}\beta = 2\upsilon^2 - 2\upsilon + 1/2 + (x^2 + x - 1/2)(\beta\bar{a}_j) \qquad (13)$$

is positive. Note that we have used the fact that $\alpha a_j = x+1$.

We will first derive an expression for the $(x^2 + x - 1/2)(\beta\bar{a}_j)$ term. Recall that

$$\beta = \frac{1}{\bar{a}_j} + \frac{1}{\bar{a}_{-j}} + \frac{2}{s}.$$

We can substitute relation (11) to obtain

$$\beta = \frac{s\underline{a}(x+1) + \bar{a}_{-j}\underline{a}(2x+1) + \bar{a}_{-j}s}{x\bar{a}_{-j}s\underline{a}},$$

while we also note that

$$\bar{a}_j = x\left[\frac{1}{\bar{a}_{-j}} + \frac{1}{\underline{a}} + \frac{1}{s}\right]^{-1} = \frac{x\bar{a}_{-j}\underline{a}s}{\bar{a}_{-j}\underline{a} + \underline{a}s + \bar{a}_{-j}s}.$$

Therefore

$$\beta\bar{a}_j = \frac{s\underline{a}(x+1) + \bar{a}_{-j}\underline{a}(2x+1) + \bar{a}_{-j}s}{\bar{a}_{-j}\underline{a} + \underline{a}s + \bar{a}_{-j}s}.$$

Later on it will be useful to have this expression with a denominator of $(\bar{a}_{-j}\underline{a} + \underline{a}s + \bar{a}_{-j}s)^2$. By multilplying numerator and denominator by a common expression we obtain

$$\beta\bar{a}_j = \frac{\begin{pmatrix} (\underline{a}^2[2x+1] + s^2 + s\underline{a}[2x+2])\bar{a}_{-j}^2 + \\ (s^2\underline{a}[x+2] + s\underline{a}^2[3x+2])\bar{a}_{-j} + \\ \underline{a}^2 s^2[x+1] \end{pmatrix}}{(\bar{a}_{-j}\underline{a} + \underline{a}s + \bar{a}_{-j}s)^2}.$$

We may now write the term of (13) we seek to evaluate as

$$(x^2+x-.5)\beta\bar{a}_j = \frac{\begin{pmatrix} \begin{pmatrix} s^2[x^2+x-.5] + s\underline{a}[2x^3+4x^2+x-1] + \\ \underline{a}^2[2x^3+3x^2-.5] \end{pmatrix}\bar{a}_{-j}^2 + \\ \begin{pmatrix} s^2\underline{a}[x^3+3x^2+1.5x-1] + \\ s\underline{a}^2[3x^3+5x^2+0.5x-1] \end{pmatrix}\bar{a}_{-j} + \\ \underline{a}^2 s^2[x^3+2x^2+.5x-.5] \end{pmatrix}}{(\bar{a}_{-j}\underline{a} + \underline{a}s + \bar{a}_{-j}s)^2}. \qquad (14)$$

We now turn to deriving an expression for the $2\upsilon^2 - 2\upsilon + 1/2$ term of (13). Recall

$$\upsilon = \frac{(2x+1)x}{2\underline{a}}\left[\frac{1}{\bar{a}_{-j}} + \frac{1}{\underline{a}} + \frac{1}{s}\right]^{-1}.$$

This reduces to

$$\upsilon = \frac{(x+1/2)xs\bar{a}_{-j}}{\bar{a}_{-j}\underline{a} + \underline{a}s + \bar{a}_{-j}s}. \qquad (15)$$

In order to have the denominator the same as that of expression (14) we multiply numerator and denominator by a common expression to obtain

$$\upsilon = \frac{s(x+1/2)x(s+\underline{a})\bar{a}_{-j}^2 + s^2\underline{a}(x+1/2)x\bar{a}_{-j}}{(\bar{a}_{-j}\underline{a} + \underline{a}s + \bar{a}_{-j}s)^2}. \qquad (16)$$

While squaring (15) yields

$$\upsilon^2 = \frac{s^2[x^4 + x^3 + x^2/4]\bar{a}_{-j}^2}{(\bar{a}_{-j}\underline{a} + \underline{a}s + \bar{a}_{-j}s)^2} \qquad (17)$$

Using expressions (16) and (17) we have that

$$\begin{aligned} & 2\upsilon^2 - 2\upsilon + .5 \\ &= \frac{\begin{pmatrix} \begin{pmatrix} s^2[2x^4+2x^3-1.5x^2-x] + \\ s\underline{a}[-2x^2-x] \end{pmatrix}\bar{a}_{-j}^2 + \\ s^2\underline{a}(-2x^2-x)\bar{a}_{-j} \end{pmatrix}}{(\bar{a}_{-j}\underline{a} + \underline{a}s + \bar{a}_{-j}s)^2} + .5 \\ &= \frac{\begin{pmatrix} \begin{pmatrix} s^2[2x^4+2x^3-1.5x^2-x+.5] + \\ s\underline{a}[-2x^2-x+1] + .5\underline{a}^2 \end{pmatrix}\bar{a}_{-j}^2 + \\ (s^2\underline{a}[-2x^2-x+1] + s\underline{a}^2)\bar{a}_{-j} + .5s^2\underline{a}^2 \end{pmatrix}}{(\bar{a}_{-j}\underline{a} + \underline{a}s + \bar{a}_{-j}s)^2} \end{aligned} \qquad (18)$$

By combining (14) and (18) we obtain

$$\lambda_{jj}\beta = \frac{\left(\begin{array}{c}\left(\begin{array}{c}s^2[2x^4+2x^3-.5x^2]+s\underline{a}[2x^3+2x^2]+\\ \underline{a}^2[2x^3+3x^2]\end{array}\right)\bar{a}_{-j}^2+\\ \\ \left(\begin{array}{c}s^2\underline{a}[x^3+x^2+.5x]+\\ s\underline{a}^2[3x^3+5x^2+0.5x]\end{array}\right)\bar{a}_{-j}+\\ \\ \underline{a}^2s^2[x^3+2x^2+.5x]\end{array}\right)}{(\bar{a}_{-j}\underline{a}+\underline{a}s+\bar{a}_{-j}s)^2}.$$

We see that the only term that contains a minus sign in the above expression is $2x^4+2x^3-.5x^2$, but this is positive for $x>0$. Therefore we may conclude that λ_{jj} is positive. ■

III. Convex Latency Functions

In this section we extend the result of Theorem 2 to convex latency functions. For convex latency functions, it is possible that a pure strategy Nash equilibrium does not exist. The authors of [1] provide such an example. Therefore, our theorem applies only in the case that a pure strategy Nash equilibrium exist.

Theorem 3: Consider the pricing game $\mathcal{G}$ with convex and differentiable latency functions. Provided that $\mathcal{G}$ has a pure strategy Nash equilibrium, the price of anarchy defined as $\frac{S(\mathcal{G})}{N(\mathcal{G})}$, is at most 1.5.

Proof: Let F be a Nash equilibrium flow vector for $\mathcal{G}$. If there is are more than one pure strategy Nash equilibria, arbitrarily pick one. Let $a_i = \frac{d}{dx}l_i(f_i)$ for each $i \in 1,...,n$, then p_i and f_i must satisfy the first order conditions laid out in Lemma 2. Thus equation 2 must hold for all $i \in 1,...n$. Let $b_i' = l(f_i) - a_i f_i$, $i \in 1,...,n$. Some b_i' might be negative, so let B be the magnitude of the largest negative b_i' term, and let $b_i = b_i' + B$ for all $i = 1,...,n$. Now consider a new game $\mathcal{G}_+$ derived from game $\mathcal{G}$ by taking

$$\begin{aligned} l_{i+}(x) &\triangleq l_i(x) + B \quad \forall\, i \in 1,...,n \text{ and } x \in \mathbb{R}^+,\\ U_+(x) &\triangleq U(x) + B \quad \forall\, x \in \mathbb{R}^+. \end{aligned}$$

It is straightforward to see that the Nash equilibrium flow, provider welfare, and user welfare should all be the same in $\mathcal{G}_+$ as in $\mathcal{G}$.

Define a new game $\mathcal{G}_{l+}$ by taking

$$\begin{aligned} l_{il+}(x) &\triangleq a_i x + b_i \quad \forall\, i \in 1,...,n \text{ and } x \in \mathbb{R}^+\\ U_+(x) &\triangleq U(x) + B \quad \forall\, x \in \mathbb{R}^+. \end{aligned}$$

The flow F is also the Nash equilibrium of the game $\mathcal{G}_{l+}$. This is because the first order conditions that determine whether a F is a Nash equilibrium in game $\mathcal{G}_{l+}$ must also be by satisfied by F to be a Nash equilibrium of $\mathcal{G}_+$. Furthermore, the provider profit and user welfare are the same as in the Nash equilibrium of $\mathcal{G}_+$.

The social optimum welfare of $\mathcal{G}_{l+}$ is not worse than the social optimum welfare of $\mathcal{G}_+$. This is because the convex latency functions of $\mathcal{G}_{l+}$ are never smaller than their tangents, which were used to make the latency functions of $\mathcal{G}_+$. Thus the optimum welfare for $\mathcal{G}_{l+}$ cannot be less than for $\mathcal{G}_+$.

Thus the price of anarchy of $\mathcal{G}$ is the same as that of $\mathcal{G}_+$. Game $\mathcal{G}_{l+}$ has the same Nash welfare as $\mathcal{G}_+$, but has a higher (not smaller) social optimal welfare. Therefore the price of anarchy of $\mathcal{G}_{l+}$ is as large as the price of anarchy of $\mathcal{G}$. Therefore 1.5, the worst case price of anarchy for games with linear latency functions, is also the worst case price of anarchy for games with convex latency functions, provided the latter has a pure strategy Nash equilibrium. ■

IV. Conclusion

We have developed a new technique for proving a bound on the price of anarchy for a network pricing game with competition, congestion, and elastic demand. We believe that this technique will be useful in studying a number of related models and extensions. One particular case that we hope to study in future work is one in which users vary in the relative values they place on price and latency. In other words, in the model of the current paper, all users only consider the sum of price and latency. An extension would be to study a model where users look at a weighted sum of the two, and the weights would be different values for different types of users.

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Correction to The Price of Anarchy in a Network Pricing Game

John Musacchio and Shuang Wu

This document corrects an error in the proof of Theorem 2 of the paper "The Price of Anarchy in a Network Pricing Game," which was presented at the Allerton Conference on Communication and Control in September 2007 [1] [1]. The error in the proof of Theorem 2 was in the last step that claims that the polynomial $2x^4 + 2x^3 - .5x^2$ is positive for $x \geq 0$, which is not true. The polynomial takes negative values between $x = 0$ and $x = \frac{1}{2}(\sqrt{2}-1)$ reaching a minimum of about -0.00354. The original proof required finding relations between the Nash equilibrium flow on links that are not used in the social optimum configuration to the flow on other links. Revising this portion of the original proof would have required revising many of the preceding steps, and probably adding even more complexity in order to retain tighter bounds through each step. Rather than taking this approach, we have instead developed a revised argument that avoids having to treat separately the links that are not used in the social optimum configuration.

In the paper, we explained why some links that are used in Nash equilibrium may not be used when prices are set to socially optimal levels. This is because the Wardrop equilibrium disutility level d^* may fall below the minimum latency b_i of one or more links i. To account for this in the original proof of Theorem 2, we separated the links into two classes, those that are used in the social optimum configuration, and those that are not. Furthermore, in our circuit analogy, we placed diode elements in each branch to prevent the flow on some branches from going in the reverse direction which would happen if b_i was larger than d^*, the common voltage seen by all the branches. We call such branches whose minimum latency is more than d^* as "undercut." Recall that the social optimum configuration flow rates are modeled by the circuit pictured in Figure 1 with the following modification. To model social optimum pricing, the resistors representing the ratio of price to flow are changed to be a_i, not $a_i + \delta_i$ which models Nash equilibrium pricing and is what is shown in the figure. The profit on each link is $a_i f_i^{*2}$, and the user welfare is $\frac{1}{2}s(\sum f_i^*)^2$ for a linear disutility function. (Recall our measure of social welfare is consumer surplus, which is found by integrating the area under the demand (disutility) function above the market clearing disutility d^*. For a linear disutility function, the region is triangular shaped, hence the factor of $\frac{1}{2}$ in the expression for user welfare.) Also note that each resistor a_i representing provider pricing is paired on the same branch with another resistor of size a_i representing that provider's latency function. Thus the social welfare (sum of profits and user welfare) in this situation is exactly half the power dissipated by resistors in the circuit.

Suppose that in the social optimum configuration we do indeed have "undercut" branches whose flow in the circuit analogy is kept from going negative (backwards) by the diode elements. A natural question is whether if the undercut branches were turned "on." (i.e. the diode preventing backward flow were removed) would the power dissipated by resistors go up or down. If one can show that the power never goes down, then one can upper-bound the social welfare of the social optimum configuration by half the power dissipated by resistors of the circuit with the diodes removed. Such a bound would greatly simplify the analysis by obviating the need of keeping track of the undercut branches. In the following lemma, we show that such an upper-bound can be constructed.

[1] We are very grateful to Xingang Shi of the Chinese University of Hong Kong for finding the error and bringing it our attention. His discovery of this problem led us to develop the corrected and improved proof we describe here.

Lemma A. *Let W_l^* be the social welfare of the social optimum configuration for a linear disutility, (thus equal to half the power dissipated by resistors in the circuit with diodes that shuts off undercut links.) Let $\widetilde{W}_l^*$ be half the power dissipated by resistors in the same circuit but with such diodes removed. Then*

$$W_l^* \le \widetilde{W}_l^*.$$

Proof: Consider the circuit representing social optimum configuration with branches $i \in \{1...m\}$ carrying positive flow. Define

$$\bar{a} = \left[\sum_{i=1}^{m} \frac{1}{a_i}\right]^{-1}, \quad \bar{b} = \bar{a}\sum_{i=1}^{m} \frac{b_i}{a_i}$$

to be the Thevinin equivalent resistance and voltage respectively of branches that carry flow in social optimum. Also note that

$$d^* = V\frac{2\bar{a}}{2\bar{a}+s} + \bar{b}\frac{s}{2\bar{a}+s}.$$

Now suppose an undercut branch $j \notin \{1..m\}$ with $b_j > d^*$ is "turned on." A current h will flow from the branch into the node shared by all the other branches. To balance the new incoming current, Kirkoff's laws dictate that the flow on the branches representing links will increase (which one can think of as one Thevenin equivalent branch) while the flow coming from source V will decrease. Because by the superposition principal this new current distributes itself according to the ratio of the conductances, source V's contributed power is reduced by

$$Vh\frac{2\bar{a}}{2\bar{a}+s}.$$

Similarly the power dissipated by the b_i voltage sources will increase by a total of

$$\bar{b}h\frac{s}{2\bar{a}+s}.$$

Since newly "turned on" voltage source b_j is larger than d^* we can use the expression for d^* to determine that the power that source b_j contributes is larger than

$$Vh\frac{2\bar{a}}{2\bar{a}+s} + \bar{b}h\frac{s}{2\bar{a}+s}.$$

Thus the power that source b_j contributes more than makes up for both the decline in power by source V and increased consumption of the b_i sources. Consequently the power dissipated by resistors in the circuit increases when undercut branch j is "turned on."

If we have several undercut branches, we can turn them on one at a time in order of increasing b_j. As we do this, we can see that the new voltage d' (the voltage of the common node that was originally d^* before turning on undercut branches) will never exceed the last b_j turned on. Consequently, each turned on branch will only increase the power dissipated by resistors. We conclude that the power dissipated by resistors in the circuit representing social optimum configuration increases (does not decrease) when all the diode elements are removed.

■

The following Lemma is similar to Theorem 1 of the original paper, but uses the simplifications made possible by Lemma A.

Lemma B. *Consider the pricing game $\mathcal{G}_t$ with linear latency functions, and a truncated linear disutility function. The difference between three times the Nash welfare $W_t = N(\mathcal{G}_t)$ and two times the social welfare $W_t^* = S(\mathcal{G}_t)$ can be expressed as*

$$3W_t - 2W_t^* \ge F^T[A + \Delta - \Delta(2A + sM)^{-1}\Delta]F. \quad (1)$$

Proof: From analyzing the voltage drops across the branches of the circuit describing the Nash equilibrium of the game we have that

$$(2A + sM + \Delta)F = \mathbf{V} - b$$

where recall from the original paper that $\mathbf{V}$ is a vector of all V's, b is a vector containing the values of each b_i, A is a diagonal matrix with entries $\{a_i\}$, M is a matrix fully populated with 1's, and Δ is a diagonal matrix with diagonal entries $\{\delta_i\}$ defined by (3) of the original paper. With a linear (not truncated) disutility curve, the total social welfare gathered in Nash equilibrium is given by

$$W = F^T(A + \frac{1}{2}sM + \Delta)F,$$

while the welfare gathered by the users is $\frac{s}{2}F^TMF$. Therefore the Nash equilibrium social welfare for a truncated linear demand curve is

$$W_t = F^T(A + \Delta)F.$$

Now consider the power dissipated by resistors in the circuit describing the social optimum configuration for linear disutility, but with the diodes removed (undercut branches "turned on."). Half of this power, which we already assigned the notation $\widetilde{W_l^*}$ can be expressed as

$$\widetilde{W_l^*} = \frac{1}{2}(\mathbf{V} - b)^T(2A + sM)^{-1}(\mathbf{V} - b).$$

By Lemma A, $W_l^* \leq \widetilde{W_l^*}$. Consequently for a linear truncated disutility function, the social welfare in the optimal configuration satisfies

$$W_t^* = W_l^* - \frac{s}{2}F^TMF \leq \tilde{W}_l^* - \frac{s}{2}F^TMF.$$

For convenience let the right most expression be denoted $\widetilde{W_t^*}$. Thus $3W_t - 2W_t^* \geq 3W_t - 2\widetilde{W_t^*}$. Combining the above identities we have that

$$\begin{aligned}
3W_t - 2\widetilde{W_t^*} &= F^T(3A + 3\Delta + sM)F - F^T(2A + sM + \Delta)(2A + sM)^{-1}(2A + sM + \Delta)F \\
&= F^T(A + 2\Delta)F - F^T(2A + sM + \Delta)(2A + sM)^{-1}\Delta F \\
&= F^T[A + \Delta - \Delta(2A + sM)^{-1}\Delta]F.
\end{aligned}$$

■

The following is a restatement of Theorem 2 of the original paper along with the new proof.

Theorem 2. *Consider the pricing game $\mathcal{G}$ with affine latency functions. The price of anarchy, defined as $\frac{S(\mathcal{G})}{N(\mathcal{G})}$ is at most 1.5.*

Proof: By Lemma 6 of the paper it is sufficient to show that the price of anarchy for a game with a truncated linear disutility function is 1.5. Without loss of generality, we may assume that all links are used in Nash equilibrium, because if otherwise, a link not used in Nash equilibrium would not be used in social optimum. Therefore one can discard the links unused in Nash equilibrium from the game without changing the price of anarchy.

From Lemma B we have that

$$3W_t - 2W_t^* \geq F^T[A + \Delta - \Delta(2A + sM)^{-1}\Delta]F. \tag{2}$$

Since the Nash equilibrium flow vector has only positive entries, it is sufficient to show that the matrix in square brackets has nonnegative entries in order to show the whole expression is nonnegative. That is the focus of the remainder of the proof. By the Matrix Inversion Lemma,

$$(2A + sM)^{-1} = \frac{1}{2}\left[A^{-1} - \frac{1}{t + 2/s}A^{-1}MA^{-1}\right]$$

where $t \triangleq \mathrm{tr}(A^{-1})$. From the definition of Δ

$$\Delta^{-1} = \left[\mathrm{tr}(A^{-1}) + \frac{1}{s}\right]I - A^{-1} = \alpha I - A^{-1}$$

and therefore $\Delta = (\alpha A - I)^{-1}A$. Substituting this relation we have that

$$\Delta(2A + sM)^{-1}\Delta = (\alpha A - I)^{-1}\left[\frac{1}{2}A - \frac{1/2}{t + 2/s}M\right](\alpha A - I)^{-1}$$

and therefore the right side of (2) can be written as

$$A + \Delta - \Delta(2A + sM)^{-1}\Delta = (\alpha A - I)^{-1}\left[\alpha^2 A^3 - \alpha A^2 - \frac{1}{2}A + \frac{1/2}{t + 2/s}M\right](\alpha A - I)^{-1}.$$

The matrix $(\alpha A - I)^{-1}$ is diagonal with positive diagonal entries. Thus to show that the matrix expression on the right side of of the above equation has nonnegative entries, it is sufficient to show that the term in square brackets has nonnegative entries. Because $\alpha = t + 1/s$ The term in square brackets is

$$\frac{1}{t + 2/s}\left[(t + 2/s)(t + 1/s)^2 A^3 - (t + 2/s)(t + 1/s)A^2 - \frac{1}{2}(t + 2/s)A + \frac{1}{2}M\right].$$

Let Ψ denote the matrix in square brackets of the above expression. The ith diagonal entry of Ψ satisfies

$$\begin{aligned}
\Psi_{ii} &= \frac{1}{t + 2/s}\left[(t + 2s^{-1})(t + 1/s)^2 a_i^3 - (t + 2/s)(t + 1/s)a_i^2 - \frac{1}{2}(t + 2/s)a_i + \frac{1}{2}\right] \\
&= \frac{1}{t + 2/s}\left[(t^3 + 4t^2 s^{-1} + 5ts^{-2} + 2s^{-3})a_i^3 - (t^2 + 3ts^{-1} + s^{-2})a_i^2 - \frac{1}{2}ta_i - a_i s^{-1} + \frac{1}{2}\right] \\
&= \frac{1}{t + 2/s}\left[(t^3 a_i^3 - t^2 a_i^2) + \left[(3t^2 a_i^2 - 3ta_i) + (t^2 a_i^2 - 1)\right] a_i s^{-1} + (5ta_i - 1)a_i^2 s^{-2} + 2a_i^3 s^{-3} - \frac{1}{2}ta_i + \frac{1}{2}\right] \\
&\geq \frac{1}{t + 2/s}\left[(ta_i)^3 - (ta_i)^2 - \frac{1}{2}(ta_i) + \frac{1}{2}\right] \\
&\geq \frac{1}{2(t + 2/s)}[ta_i - 1]
\end{aligned}$$

Note the last two steps use the fact that $ta_i \geq 1$. Consequently, we have that

$$2(t + 2/s)\Psi \geq \begin{bmatrix} ta_1 - 1 & 1 & \dots & 1 \\ 1 & ta_2 - 1 & \dots & 1 \\ \vdots & \vdots & \dots & \vdots \\ 1 & 1 & \dots & ta_N - 1 \end{bmatrix}$$

where the $\geq$ relation is element by element. The right side has all nonnegative entries, therefore Ψ, and consequently $[A + \Delta - \Delta(2A + sM)^{-1}\Delta]$ have all nonnegative entries. Thus $3W_t - 2W_t^* \geq 0$.

■

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